FIXED-DEGREE ATKIN–LEHNER SIGN STATISTICS FOR HILBERT NEWFORMS

Abstract. We conjecture joint equidistribution of prescribed local Atkin–Lehner signs among genuine parallel-weight-two Hilbert newform orbits of a fixed Hecke-field degree. Ambient Plancherel equidistribution gives the corresponding statement without the rationality-degree restriction. The fixed-degree condition is global and thin, and no known trace-formula projector isolates it.

1. The fixed-degree family

Let F be a real quadratic field with narrow class number one, let $d \geq 1$, and let S be a finite nonempty set of prime ideals of $\mathcal{O}_F$ satisfying $\mathfrak{q} \nmid 2D_F$ for every $\mathfrak{q} \in S$. Let $\mathcal{H}_{F,d,S}(X)$ be the Hecke-Galois orbits O of parallel-weight-two Hilbert newforms such that

- the central character is trivial;
- the form is non-CM and not a base change from $\mathbb{Q}$;
- $[K_O : \mathbb{Q}] = d$;
- $\mathrm{N}\mathfrak{n}_O \leq X$;
- $v_{\mathfrak{q}}(\mathfrak{n}_O) = 1$ for every $\mathfrak{q} \in S$.

Write $W_{\mathfrak{q}}(O) \in \{\pm 1\}$ for the local Atkin–Lehner eigenvalue.

Conjecture 1. *If $\#\mathcal{H}_{F,d,S}(X) \to \infty$, then for every $\eta = (\eta_{\mathfrak{q}})_{\mathfrak{q} \in S} \in \{\pm 1\}^S$,*

$$\lim_{X \to \infty} \frac{\#\{O \in \mathcal{H}_{F,d,S}(X) : W_{\mathfrak{q}}(O) = \eta_{\mathfrak{q}} \text{ for all } \mathfrak{q} \in S\}}{\#\mathcal{H}_{F,d,S}(X)} = 2^{-|S|}.$$

2. Numerical evidence

The available data contain 75268 genuine orbits with 109751 reliable local observations at primes of exact level valuation one. The largest complete singleton family is

$$F = \mathbb{Q}(\sqrt{2}), \qquad d = 1,$$

at the inert prime of norm 9: the two signs occur 604 and 608 times. Over $\mathbb{Q}(\sqrt{5})$, the analogous degree-one count is 334 versus 304.

Several two-prime samples are shown in Table 1; columns correspond to the four sign vectors in a fixed order.

The finite samples are compatible with uniformity, but meaningful tests for larger sets S, large fixed degrees, and many base fields are not yet available.

Key words and phrases. Hilbert modular forms, Atkin–Lehner signs, root numbers, Hecke fields, fields of rationality, equidistribution.

family	$++$	$+-$	$-+$	$--$
$\mathbb{Q}(\sqrt{5})$, selected primes of norms 4 and 5, all degrees	141	163	164	165
$\mathbb{Q}(\sqrt{5})$, same pair, degree 1	59	77	84	73
$\mathbb{Q}(\sqrt{5})$, primes of norms 4 and 11	87	76	82	89
$\mathbb{Q}(\sqrt{5})$, primes of norms 5 and 11	64	65	59	64
$\mathbb{Q}(\sqrt{5})$, latter pair, degree 1	29	31	31	29
$\mathbb{Q}(\sqrt{2})$, degree 1, inert norms 25 and 9	12	16	20	12

TABLE 1. Selected joint local-sign counts.

3. Fourier reformulation

Put

$$N_d(X) = \#\mathcal{H}_{F,d,S}(X)$$

and, for $T \subseteq S$, define the sign correlation

$$C_T(X) = \sum_{O \in \mathcal{H}_{F,d,S}(X)} \prod_{\mathfrak{q} \in T} W_{\mathfrak{q}}(O).$$

The identity

$$\mathbf{1}_{\{W(O)=\eta\}} = 2^{-|S|} \sum_{T \subseteq S} \prod_{\mathfrak{q} \in T} \eta_{\mathfrak{q}} W_{\mathfrak{q}}(O)$$

shows that Conjecture 1 is equivalent to

$$C_T(X) = o(N_d(X)) \qquad \text{for every nonempty } T \subseteq S.$$

Atkin–Lehner signs are constant on Galois orbits. Since every orbit in the family has degree d, passing from orbit sums to sums over conjugate eigenforms multiplies both C_T and N_d by d; the difficulty is therefore not the orbit weighting itself.

4. Local Plancherel heuristic

At a prime $\mathfrak{q} \nmid 2D_F$ of conductor exponent one and trivial central character, the local representation is an unramified quadratic twist of the Steinberg representation. The two possibilities have opposite Atkin–Lehner signs and equal local Plancherel mass. Exact-new Plancherel equidistribution consequently predicts independent uniform signs in the ambient spectrum.

The exact rationality-degree condition is global and is not the eigenspace of a known trace-formula operator. Bounded-degree forms have density zero in the full spectrum, so ambient equidistribution does not determine their internal sign distribution. Excluding CM and base-change forms imposes two further global restrictions.

5. Why twisting does not prove the result

A quadratic twist can flip prescribed local Steinberg signs while preserving the Hecke field, but generally introduces auxiliary conductor. It may send levels bounded by X to levels bounded by cX rather than preserving the same cutoff, and it need not preserve the literal non-base-change family. A bijection between two differently scaled height ranges does not imply equality of same-cutoff densities without regular variation of the counting function. Narrow class number one also eliminates a general supply of nontrivial everywhere-unramified quadratic characters that would give a level-preserving involution.

6. Main obstacles

A proof requires fixed-degree correlation estimates

$$\sum_{O\in\mathcal{H}_{F,d,S}(X)}\prod_{\mathfrak{q}\in T} W_{\mathfrak{q}}(O)=o\big(\#\mathcal{H}_{F,d,S}(X)\big)$$

for every nonempty T, after removing CM and base-change orbits. No asymptotic count is known for the denominator itself, and no trace formula isolates exact Hecke-field degree. The analogous question is open even for classical weight-two newforms of a fixed rationality degree at varying prime level. A refined version of the conjecture may need to condition on the complete local representation type at primes dividing the discriminant of F or at other systematically ramified places.

References

[1] J. Binder, *Fields of rationality of cusp forms*, Israel J. Math. **222** (2017), 973–1028.
[2] R. Schmidt, *Some remarks on local newforms for* GL(2), J. Ramanujan Math. Soc. **17** (2002), 115–147.
[3] K. Martin, *Refined dimensions of cusp forms, and equidistribution and bias of signs*, J. Number Theory **188** (2018), 1–17.
[4] Z. Luo, Q. Pi and H. Wu, *Bias of root numbers for Hilbert newforms of cubic level*, J. Number Theory **247** (2023), 378–409.
[5] L. Dieulefait, A. Pacetti and P. Tsaknias, *On the number of Galois orbits of newforms*, J. Eur. Math. Soc. **23** (2021), 2833–2860.
[6] J. Weinstein, *Hilbert modular forms with prescribed ramification*, Int. Math. Res. Not. IMRN (2009), 1388–1420.