SQUAREFREE-DISCRIMINANT BIAS IN UNIT SIGNATURES OF S_5-QUINTIC FIELDS

Abstract. We record a conjectural ramification bias in the unit-signature ranks of discriminant-ordered S_5-quintic fields. After fixing the signature and the exact set of common index divisors, fields of squarefree discriminant appear more likely to have a deficient unit-signature rank than fields of nonsquarefree discriminant. The observed inequalities are stable in the largest available height shells, but no existing random-space model predicts their strict direction.

1. Statement of the bias

Let $K/\mathbb{Q}$ be an S_5-quintic field and write

$$s(K) = \operatorname{sgnrk}(K).$$

For $r_2 \in \{0,1\}$, $\eta \in \{\mathrm{sf}, \mathrm{nsf}\}$, and $T \subseteq \{2,3\}$, let $\mathcal{F}_{r_2,\eta,T}(X)$ be the family defined by signature, squarefreeness, common-index set, and the bound $|D_K| \leq X$.

Conjecture 1. *For every T for which the relevant strata are infinite, all displayed limits exist and*

$$\begin{aligned}
\lim_{X\to\infty} \Pr_{\mathcal{F}_{0,\mathrm{sf},T}(X)}(s \leq 4) &> \lim_{X\to\infty} \Pr_{\mathcal{F}_{0,\mathrm{nsf},T}(X)}(s \leq 4),\\
\lim_{X\to\infty} \Pr_{\mathcal{F}_{0,\mathrm{sf},T}(X)}(s \leq 3) &> \lim_{X\to\infty} \Pr_{\mathcal{F}_{0,\mathrm{nsf},T}(X)}(s \leq 3),\\
\lim_{X\to\infty} \Pr_{\mathcal{F}_{1,\mathrm{sf},T}(X)}(s \leq 2) &> \lim_{X\to\infty} \Pr_{\mathcal{F}_{1,\mathrm{nsf},T}(X)}(s \leq 2).
\end{aligned}$$

The first two inequalities concern totally real fields, while the third concerns signature $(3,1)$. Through

$$\frac{h_K^+}{h_K} = 2^{r_1 - s(K)},$$

they are equivalent to strict squarefree biases toward larger narrow-to-ordinary class-number ratios.

2. Numerical evidence

The reliable census contains 162022 totally real fields and 225089 fields of signature $(3,1)$. The terminal probabilities are shown in Table 1.

Every comparison has the predicted sign. The same direction persists in the upper halves of the complete windows and, after initial fluctuations, at

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	$(5,0): s \le 4$		$(5,0): s \le 3$		$(3,1): s \le 2$	
T	sf	nsf	sf	nsf	sf	nsf
$\varnothing$	0.631835	0.578511	0.055831	0.036575	0.283043	0.231891
$\{2\}$	0.612838	0.571893	0.049020	0.035108	0.271858	0.223093
$\{3\}$	0.582746	0.554585	0.058099	0.021834	0.307412	0.248538

Table 1. Terminal squarefree and nonsquarefree signature-deficiency probabilities.

all later cumulative cutoffs. The $T = \{3\}$ samples are substantially smaller; in one upper-shell comparison the difference is only about 0.0025, so the apparent universality across all T is not yet strongly tested.

3. Possible arithmetic mechanism

Squarefree discriminant changes the local ramification ensemble at every prime. Unit signatures are controlled by the global position of units inside a 2-Selmer signature space, so an indirect correlation with ramification is possible. The persistence of the bias after fixing $I(K) = T$ indicates that it is not explained solely by the principal small-prime obstruction to monogenicity.

There is, however, no established heuristic that predicts the strict direction in Conjecture 1. A naive extension of local-condition independence might instead predict equality of the limiting signature distributions after fixing the signature. The conjecture therefore asserts a genuinely new correlation, not merely a refinement of the standard Dummit–Voight model.

4. Relation with known results

Dummit and Voight conjecture the unconditioned unit-signature distribution of odd-degree S_n-fields and prove the maximal-isotropic structure of the associated 2-Selmer signature image. Bhargava's geometric sieve gives positive linear asymptotics for squarefree-discriminant S_5-fields with acceptable local specifications. The Armitage–Fröhlich inequalities relate signature deficiency to ordinary and narrow 2-class ranks. None of these results controls the correlation between squarefreeness and the position of the unit subspace.

The denominator families are therefore well motivated and, under local counting, substantial. The missing object is a marked count by signature rank.

5. Main obstacles and tests

Let $N^{\le a}_{\eta,\tau,T}(X)$ count fields of squarefree status η, signature τ, and common-index set T whose signature rank is at most a. A proof first requires existence of the ratios

$$\frac{N^{\leq a}_{\eta,\tau,T}(X)}{N_{\eta,\tau,T}(X)}$$

after which one must compare the leading constants strictly. No asymptotic is known for the numerator even without the η and T restrictions.

Several computational tests would materially strengthen the conjecture:

- compare disjoint high-discriminant shells rather than nested cumulative samples;
- stratify further by the complete local ramification type at 2 and by the number of ramified primes;
- estimate the bias after matching squarefree and nonsquarefree fields by discriminant scale and local mass;
- enlarge the sparse $T = \{3\}$ and $T = \{2, 3\}$ samples.

A stable nonzero gap after these controls would provide much stronger evidence for a true global ramification effect.

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