BOUNDED HECKE DEGREES IN GENUINE HILBERT NEWFORM FAMILIES

Abstract. Over a fixed real quadratic field of narrow class number one, we consider non-CM, non-base-change Hilbert newforms of parallel weight two and order their Hecke-Galois orbits by level norm. We conjecture that, when every orbit receives equal weight, those with bounded Hecke-field degree have density zero. This is stronger than known eigenform-weighted rationality-field results and is naturally related to generalized Maeda heuristics.

1. The orbit-counting problem

Let F be a real quadratic field with narrow class number one. For $X > 0$, let $\mathcal{H}_F(X)$ be the set of Hecke-Galois orbits of parallel-weight-two Hilbert newforms over F satisfying

- trivial central character;
- level ideal $\mathfrak{n}$ with $\mathrm{N}\mathfrak{n} \le X$;
- non-CM and not a base change from $\mathbb{Q}$.

For $O \in \mathcal{H}_F(X)$, let K_O be its Hecke field.

Conjecture 1. *For every fixed integer $D \ge 1$, if $\#\mathcal{H}_F(X) \to \infty$, then*

$$\lim_{X\to\infty} \frac{\#\{O \in \mathcal{H}_F(X) : [K_O : \mathbb{Q}] \le D\}}{\#\mathcal{H}_F(X)} = 0.$$

The measure is unweighted: a rational orbit and an orbit of degree one hundred each contribute one object.

2. Numerical evidence

After removing visibly partial bounded-degree-only tails, the available data contain 85306 genuine orbits over 42 fields. Twelve fields contain at least 1000 reliable orbits. In every one of those fields, the final bounded-degree proportion is lower than its first-quartile value for each $D \in \{1, 2, 4, 8, 16, 32\}$.

The two deepest fields are summarized in Table 1.

Across the twelve substantial fields, the mean degree-one proportion falls from 0.3640 in the first quartile to 0.3225 at the final cutoff, while the mean degree-at-most-eight proportion falls from 0.8675 to 0.7524. The largest reliable Hecke degree is 179.

Key words and phrases. Hilbert modular forms, fields of rationality, Hecke fields, Galois orbits, density zero.

F	statistic	$X = 1000$	$X = 2000$	$X = 5000$
$\mathbb{Q}(\sqrt{5})$	$\Pr([K_O : \mathbb{Q}] = 1)$	0.449	0.383	0.317
	$\Pr([K_O : \mathbb{Q}] \leq 8)$	0.973	0.922	0.808
	mean degree	2.28	3.14	5.45
$\mathbb{Q}(\sqrt{2})$	$\Pr([K_O : \mathbb{Q}] = 1)$	0.432	0.373	0.327
	$\Pr([K_O : \mathbb{Q}] \leq 8)$	0.920	0.848	0.742
	mean degree	3.28	4.79	8.78

Table 1. Unweighted orbit statistics at increasing level-norm cutoffs.

3. Why the weighting matters

Binder's theorem implies that bounded fields of rationality have density zero when individual eigenforms are counted. In orbit language, the orbit O receives weight $[K_O : \mathbb{Q}]$. This does not imply Conjecture 1. A family could contain many bounded-degree orbits together with a small number of enormous orbits that dominate the degree-weighted mass.

Conjecture 1 asserts an escape of degree for the counting measure on irreducible Hecke factors themselves. Generalized Maeda heuristics suggest an even stronger picture in which a small number of giant orbits dominate each natural local sector.

4. Relation with known results

Binder proves bounded rationality degree has zero eigenform-weighted density for Hilbert cusp forms in the level aspect. Shin and Templier establish broad growth and finiteness principles for fields of rationality of automorphic representations. Maeda-type work on classical and Hilbert newforms predicts that, after fixing unavoidable local invariants, generic Hecke algebras have very few irreducible factors. None of these results controls the number of bounded-degree factors with each factor counted once.

The exclusion of CM and base change is essential. Both mechanisms can generate infinite low-degree families with highly structured coefficient fields. Removing them isolates the genuinely Hilbert part of the spectrum but introduces a global condition not detected by local limit multiplicity.

5. Main obstacles

A proof requires an orbit-counting theorem, not only a dimension theorem. It would be enough to show that for every fixed D,

$$\#\{O \in \mathcal{H}_F(X) : [K_O : \mathbb{Q}] \leq D\} = o(\#\mathcal{H}_F(X)).$$

At present there is no asymptotic for either side. Trace formulas count embeddings of Hecke orbits and therefore naturally produce degree-weighted statistics. Passing to unweighted orbit counts requires control of the factorization pattern of Hecke algebras, a problem of Maeda type. The finite data strongly suggest degree escape, but the observed proportions for $D = 8$ and $D = 16$ remain large enough that much deeper levels are needed to distinguish eventual decay from a slowly stabilizing positive proportion.

References

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