UNIT SIGNATURES AND COMMON INDEX DIVISORS IN S_5-QUINTIC FIELDS

Abstract. We conjecture that, after fixing the signature and the squarefree or nonsquarefree discriminant regime, the limiting unit-signature distribution of discriminant-ordered S_5-quintic fields is independent of the exact set of common index divisors. The conjecture separates an archimedean 2-primary invariant from finite local splitting conditions at 2 and 3.

1. Families and signature rank

Let $K/\mathbb{Q}$ be an S_5-quintic field of signature (r_1, r_2), and define

$$\operatorname{sgnrk}(K) = \dim_{\mathbb{F}_2} \operatorname{im}\bigl(\operatorname{sgn} : \mathcal{O}_K^\times \longrightarrow \{\pm 1\}^{r_1}\bigr).$$

For $r_2 \in \{0, 1\}$, put $r_1 = 5 - 2r_2$. The case $r_2 = 2$ has only one real place and hence a deterministic signature rank, so it is omitted.

Fix $\eta \in \{\mathrm{sf}, \mathrm{nsf}\}$ and $T \subseteq \{2, 3\}$. Let $\mathcal{F}_{r_2,\eta,T}(X)$ be the $\mathbb{Q}$-isomorphism classes of S_5-quintic fields with

$$|D_K| \le X, \qquad I(K) = T,$$

and with squarefree discriminant precisely when $\eta = \mathrm{sf}$.

Conjecture 1. *For every $r_2 \in \{0, 1\}$ and $\eta \in \{\mathrm{sf}, \mathrm{nsf}\}$, there is a probability measure $\nu_{r_2,\eta}$ on $\{1, \dots, r_1\}$ such that, for every T for which the stratum is infinite and every s,*

$$\lim_{X\to\infty} \Pr_{K\in\mathcal{F}_{r_2,\eta,T}(X)} \bigl(\operatorname{sgnrk}(K) = s\bigr) = \nu_{r_2,\eta}(s).$$

In particular, the limiting distribution is independent of T.

The measure is allowed to depend on squarefreeness. Thus this conjecture is compatible with a separate ramification bias while asserting independence from the exact common-index obstruction.

2. Numerical evidence

The reliable windows contain 162022 totally real fields and 225089 fields of signature $(3, 1)$. Table 1 compares the full-signature-rank frequency in the upper halves of the two largest common-index strata.

The four-shell trajectories for $T = \varnothing$ and $T = \{2\}$ show broadly parallel height drift. The $T = \{3\}$ samples contain only a few hundred fields in the upper shells and fluctuate by several percentage points in both directions.

Key words and phrases. quintic fields, unit signatures, common index divisors, narrow class groups, local-global independence.

signature	discriminant	$T = \varnothing$	$T = \{2\}$	difference
$(5, 0)$	nsf	41.253%	42.287%	1.034 pp
$(5, 0)$	sf	36.092%	38.182%	2.090 pp
$(3, 1)$	nsf	75.457%	76.741%	1.284 pp
$(3, 1)$	sf	70.963%	71.395%	0.432 pp

Table 1. Upper-shell full-signature-rank frequencies.

They do not yet distinguish a genuine limiting shift from sampling noise. The $T = \{2, 3\}$ stratum is absent from the reliable finite windows.

3. Heuristic motivation

The unit-signature rank is an archimedean component of the 2-Selmer signature map. By contrast, the condition $I(K) = T$ is determined by the splitting behavior at the finite primes 2 and 3. Once the broader ramification regime is fixed by squarefreeness or nonsquarefreeness, a product law between these archimedean and finite local data is a natural first model.

The fieldwise identity

$$\frac{h_K^+}{h_K} = 2^{r_1 - \operatorname{sgnrk}(K)}$$

links signature deficiency to the difference between narrow and ordinary class groups. It constrains the possible values of $\operatorname{sgnrk}(K)$ but does not force a dependence on the common-index set.

4. Relation with known results

Dummit and Voight conjecture the complete unconditioned unit-signature distribution for odd-degree S_n-fields of fixed signature. Their model is based on the maximal totally isotropic image of the 2-Selmer signature map. Bhargava's geometric sieve and local counting theorems give linear asymptotics for the underlying S_5-field strata, while the common-index criterion makes $I(K) = T$ a finite local condition at 2 and 3. None of these results counts fields marked by unit-signature rank.

The conjecture therefore has two layers: existence of each conditioned signature distribution, and equality of those distributions across T. Even the first layer is open for the unrestricted discriminant-ordered S_5 family.

5. Main obstacles

Writing $N_{r_2,\eta,T,s}(X)$ for the number of fields in the stratum with signature rank s, a proof requires

$$N_{r_2,\eta,T,s}(X) = c_{r_2,\eta,T}\, \nu_{r_2,\eta}(s) X + o(X)$$

with the same $\nu_{r_2,\eta}$ for every T. The unmarked denominator asymptotic is accessible through acceptable local specifications. The marked numerator requires conditional equidistribution of the global unit-generated subspace inside the 2-Selmer signature space after fixing squarefreeness and the completions at 2 and 3. No current counting theorem provides this joint distribution.

References

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