THE DENSITY OF $\mathbb{Q}$-CURVES OVER A FIXED QUADRATIC FIELD

Abstract. For a fixed quadratic field, we conjecture that non-CM $\mathbb{Q}$-curves have density zero among all non-CM elliptic-curve isogeny classes ordered by conductor norm. The assertion includes base changes from $\mathbb{Q}$ as well as strict $\mathbb{Q}$-curves. Numerical data over the deepest imaginary quadratic fields show a consistent decline in both components, while the main theoretical difficulty is obtaining conductor-aspect counts uniform over all modular degrees and twist families.

1. Statement

Fix a quadratic field K. Let $\mathcal{E}_K(X)$ be the set of non-CM K-isogeny classes $\mathcal{C}$ of elliptic curves satisfying

$$\mathrm{N}_{K/\mathbb{Q}}\mathfrak{N}_{\mathcal{C}} \leq X.$$

Let $\mathcal{Q}_K(X) \subseteq \mathcal{E}_K(X)$ be the subset of $\mathbb{Q}$-curve classes, namely those for which an elliptic curve in the class is isogenous over $\overline{\mathbb{Q}}$ to each of its Galois conjugates. Base changes from $\mathbb{Q}$ are included.

Conjecture 1. *For every quadratic field K,*

$$\lim_{X\to\infty} \frac{\#\mathcal{Q}_K(X)}{\#\mathcal{E}_K(X)} = 0.$$

Both the conductor and the $\mathbb{Q}$-curve property are invariant under K-isogeny, so the formulation is intrinsic to isogeny classes.

2. Numerical evidence

The complete scan contains 297834 non-CM quadratic-field isogeny classes, of which 21253 are $\mathbb{Q}$-curves: 11712 base-change classes and 9541 strict $\mathbb{Q}$-curves. In every one of the thirteen largest fixed-field datasets, the cumulative $\mathbb{Q}$-curve proportion decreases as the conductor cutoff grows. Table 1 gives the five deepest fields.

The decline persists after separating base changes and strict $\mathbb{Q}$-curves. In the five fields of Table 1, base-change shares fall from roughly 3.5%–5.6% in the first shell to 0.9%–1.6% in the last, while strict-$\mathbb{Q}$-curve shares fall to approximately 0.7%–1.0%.

Key words and phrases. $\mathbb{Q}$-curves, elliptic curves over quadratic fields, conductor ordering, base change, thin families.

D_K	first conductor decile	full range	terminal shell
-3	6.633%	3.220%	1.970%
-4	6.607%	2.920%	2.344%
-7	6.742%	2.968%	1.893%
-8	7.772%	3.471%	2.076%
-11	9.452%	3.949%	2.063%

Table 1. Decline of the $\mathbb{Q}$-curve proportion in the deepest fixed-field datasets.

3. Geometric and automorphic motivation

Being a $\mathbb{Q}$-curve is a strong descent condition. A non-CM $\mathbb{Q}$-curve is associated with a modular object carrying an inner-twist structure, and central $\mathbb{Q}$-curves of a fixed squarefree degree are parametrized by suitable Atkin–Lehner quotients or twists of modular curves. The union over all degrees is countable but highly structured; it should be thin inside the two-parameter moduli of elliptic curves over K.

The conductor ordering prevents a direct transfer of geometric height-density statements. A fixed geometric $\mathbb{Q}$-curve class has infinitely many quadratic twists, all of which remain $\mathbb{Q}$-curves. At a new good prime ramified in the twisting character, the conductor exponent is typically two. Thus a single seed already generates a family on an $X^{1/2}$ scale. Conjecture 1 predicts that the full family of elliptic curves over K grows faster.

4. Relation with known results

Ribet characterizes non-CM $\mathbb{Q}$-curves as geometric factors of abelian varieties of GL_2-type over $\mathbb{Q}$. Modular-curve and Chabauty methods determine $\mathbb{Q}$-curve loci for specified degrees, and modern computations determine quadratic points on many individual curves $X_0(N)$. These are level-by-level results. They do not give a count uniform in the degree of the isogeny to the conjugate.

There is no general asymptotic count of all elliptic curves over a quadratic field by conductor norm. Even over $\mathbb{Q}$, conductor-aspect asymptotics are known only in restricted families. Consequently, neither the numerator nor the denominator in Conjecture 1 is presently accessible at the required precision.

5. Main obstacles

A proof needs a numerator estimate that beats the expected growth of the ambient family. One possible target is

$$\#\mathcal{Q}_K(X) = o\big(X^{5/6}(\log X)^{-C_K}\big),$$

paired with a lower bound of the indicated size for non-$\mathbb{Q}$ curves. Establishing such an estimate requires uniform control over

- all squarefree central $\mathbb{Q}$-curve degrees;
- rational points on the associated Atkin–Lehner quotients and twists;
- all quadratic twists of every geometric class;
- the passage from geometric or modular height to conductor norm.

A uniform bound on possible $\mathbb{Q}$-curve degrees would not by itself solve the problem, because each surviving geometric class has infinitely many twists. Conversely, a sufficiently strong conductor-to-height inequality would permit geometric thinness results to enter, but such inequalities are tied to unresolved Szpiro/abc-type phenomena.

References

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