GOOD-PRIME CLASS GROUPS OF S_5-QUINTIC FIELDS WITH PRESCRIBED COMMON INDEX DIVISORS

Abstract. We propose a joint Cohen–Lenstra–Martinet law for the good-primary parts of class groups of discriminant-ordered S_5-quintic fields. The signature and the set of common index divisors are fixed. For primes not dividing $|S_5|$, the primary class groups are conjectured to follow the standard u-probability measure and to be asymptotically independent across distinct primes.

1. The conjectural measure

Let $K/\mathbb{Q}$ be an S_5-quintic field of signature $(5-2r_2, r_2)$, and put

$$u = 4 - r_2.$$

For $T \subseteq \{2,3\}$, let $\mathcal{F}_{r_2,T}(X)$ denote the $\mathbb{Q}$-isomorphism classes of such fields with $|D_K| \le X$ and $I(K) = T$, where $I(K)$ is the set of common index divisors.

For a prime $\ell \ge 7$ and a finite abelian ℓ-group A, define

$$\mu_{\ell,u}(A) = \frac{c_{\ell,u}}{|A|^u |\operatorname{Aut}(A)|}, \qquad c_{\ell,u} = \prod_{j=u+1}^{\infty} (1 - \ell^{-j}).$$

The Hall identity implies that $\mu_{\ell,u}$ is a probability measure.

Conjecture 1. *Assume that $\#\mathcal{F}_{r_2,T}(X) \to \infty$. For every finite set S of primes $\ell \ge 7$ and every collection of finite abelian ℓ-groups $(A_\ell)_{\ell \in S}$,*

$$\lim_{X\to\infty} \Pr_{K \in \mathcal{F}_{r_2,T}(X)} \left(\mathrm{Cl}(K)[\ell^\infty] \simeq A_\ell \text{ for all } \ell \in S\right) = \prod_{\ell \in S} \mu_{\ell,u}(A_\ell).$$

The restriction $\ell \ge 7$ removes the primes $2, 3$, and 5 dividing $|S_5| = 120$, where the naive Cohen–Lenstra–Martinet measure requires correction.

2. Numerical evidence

The reliable windows contain 885295 fields. Good-prime divisibility is very rare in signatures of large unit rank, as predicted by the factor $|A|^{-u}$. The observed counts are given in Table 1.

For $r_2 = 2$, the 7-divisibility counts by common-index set are

$$\frac{566}{423127}, \qquad \frac{128}{69654}, \qquad \frac{7}{5300}, \qquad \frac{0}{103}$$

Key words and phrases. quintic fields, class groups, Cohen–Lenstra–Martinet heuristics, common index divisors, local conditions.

r_2	fields	$7 \mid h_K$	$11 \mid h_K$	$7 \cdot 11 \mid h_K$
0	162022	0	0	0
1	225089	5	0	0
2	498184	701	76	0

TABLE 1. Good-prime divisibility in the primary discriminant windows.

for $T = \varnothing, \{2\}, \{3\}, \{2,3\}$, respectively. The conjectural nontrivial-primary probabilities for $u = 2$ are approximately

$$1 - c_{7,2} = 0.003399914, \qquad 1 - c_{11,2} = 0.000826389.$$

The current frequencies lie well below these limits. They increase substantially across successive discriminant quarters: for $r_2 = 2$, the numbers of fields with $7 \mid h_K$ are

$$65, 172, \qquad 228, \qquad 236$$

in quarters containing respectively 103380, 125475, 132514, and 136815 fields. No 49- or 121-divisibility occurs in the primary windows, and the absence of a joint 7–11 event has little statistical force because the conjectural expectation is only about 1.4.

3. Heuristic motivation

For non-Galois S_n-fields, the Cohen–Lenstra–Martinet formalism uses the augmentation representation. In the S_5/S_4 situation its relevant rank is the unit rank $u = 4 - r_2$, giving precisely the measure $\mu_{\ell,u}$ at primes $\ell \nmid 120$. Distinct good primes are expected to behave independently.

The condition $I(K) = T$ is determined by the completions at 2 and 3. It is therefore natural to expect independence between this small-prime local condition and unramified abelian ℓ-extensions for $\ell \geq 7$. Conjecture 1 is the assertion that this heuristic survives discriminant ordering and the exact common-index conditioning.

4. Relation with known results

Wang and Wood give a modern moment interpretation of the Cohen–Lenstra–Martinet distributions for non-Galois fields, and Wood formulates local-condition refinements in other settings. Bhargava's quintic counting theorem provides the underlying locally specified field counts, but not class-group statistics. The strongest theorem-level analogues concern particular torsion moments in lower-degree or specially parametrized families. No good-prime class-group distribution is known even in the unconditioned discriminant-ordered S_5-quintic family.

5. Moment formulation and obstacles

For a finite abelian ℓ-group B, the conjectural measure is characterized by the moments

$$\mathbb{E}\,\#\,\mathrm{Sur}(\mathrm{Cl}(K)[\ell^\infty], B) = |B|^{-u}.$$

Thus even the first C_7 moment requires

$$\frac{1}{\#\mathcal{F}_{r_2,T}(X)} \sum_{K\in\mathcal{F}_{r_2,T}(X)} \#\,\mathrm{Sur}(\mathrm{Cl}(K), C_7) \longrightarrow 7^{-u}.$$

By class field theory this is a count of unramified cyclic degree-seven extensions of locally specified quintic fields. Joint independence requires all mixed moments at finitely many primes, as well as tightness sufficient to recover the full distribution. Existing quintic parametrizations count the base fields but do not count these unramified towers.

References

[1] H. Cohen and J. Martinet, *Etude heuristique des groupes de classes des corps de nombres*, J. Reine Angew. Math. **404** (1990), 39–76.

[2] W. Wang and M. M. Wood, *Moments and interpretations of the Cohen–Lenstra–Martinet heuristics*, Comment. Math. Helv. **96** (2021), 339–387.

[3] M. M. Wood, *Cohen–Lenstra heuristics and local conditions*, Res. Number Theory **4** (2018), Paper No. 41.

[4] M. Bhargava, *The density of discriminants of quintic rings and fields*, Ann. of Math. **172** (2010), 1559–1591.

[5] H. T. Engstrom, *On the common index divisors of an algebraic field*, Trans. Amer. Math. Soc. **32** (1930), 223–237.

[6] A. Bartel and H. W. Lenstra Jr., *On class groups of random number fields*, Proc. Lond. Math. Soc. **121** (2020), 927–953.