GIANT HECKE ORBITS IN PRIME-LEVEL HILBERT NEWSPACES

Abstract. We formulate a level-aspect Hilbert analogue of a Maeda-type giant-orbit conjecture. Over a fixed real quadratic field of narrow class number one, and separately in each prime-level functional-equation-sign sector, the largest non-CM, non-base-change Hecke orbit is conjectured to occupy asymptotically all of the sector. The numerical evidence is strong in the deepest available fields, but the corresponding classical prime-level statement is itself open.

1. Prime-level sign sectors

Let F be a real quadratic field with narrow class number one, and let $\mathfrak{p} \nmid 2D_F$ be a prime ideal. For $\varepsilon \in \{\pm 1\}$, let $\mathcal{H}_{F,\mathfrak{p},\varepsilon}$ denote the Hecke-Galois orbits of parallel-weight-two Hilbert newforms over F with trivial character and exact level $\mathfrak{p}$, subject to the following conditions:

- the forms are non-CM;
- they are not base changes from $\mathbb{Q}$;
- their completed standard L-functions have sign ε.

For an orbit O, write K_O for its Hecke field and set

$$T_{\mathfrak{p},\varepsilon} = \sum_{O \in \mathcal{H}_{F,\mathfrak{p},\varepsilon}} [K_O : \mathbb{Q}], \qquad M_{\mathfrak{p},\varepsilon} = \max_{O \in \mathcal{H}_{F,\mathfrak{p},\varepsilon}} [K_O : \mathbb{Q}].$$

Conjecture 1. *For each $\varepsilon \in \{\pm 1\}$, the sector $\mathcal{H}_{F,\mathfrak{p},\varepsilon}$ is nonempty for all sufficiently large $\mathrm{N}\mathfrak{p}$. Moreover,*

$$\lim_{\substack{\mathrm{N}\mathfrak{p} \to \infty \\ \mathfrak{p} \nmid 2D_F}} \frac{M_{\mathfrak{p},\varepsilon}}{T_{\mathfrak{p},\varepsilon}} = 1.$$

Thus the rational Hecke module in each sign sector is conjecturally the direct sum of one constituent of dimension $(1 - o(1))T_{\mathfrak{p},\varepsilon}$ and exceptional constituents of total dimension $o(T_{\mathfrak{p},\varepsilon})$.

2. Numerical evidence

The available data contain 7717 non-CM, non-base-change orbits at 2661 represented prime-ideal levels, giving 5322 sign sectors over 42 real quadratic fields of narrow class number one. Table 1 records the mean of M/T in consecutive norm bins for the two deepest fields.

Key words and phrases. Hilbert modular forms, Hecke fields, Galois orbits, Maeda conjecture, Atkin–Lehner signs, prime level.

F	1001–2000	2001–3000	3001–4000	4001–5000
$\mathbb{Q}(\sqrt{5})$	0.8704	0.9261	0.9463	0.9659
$\mathbb{Q}(\sqrt{2})$	0.9542	0.9760	0.9920	0.9937

Table 1. Mean largest-orbit share, pooling the two sign sectors.

In the final bin for $\mathbb{Q}(\sqrt{5})$, the sign-separated means are 0.9450 for sign -1 and 0.9843 for sign $+1$. For $\mathbb{Q}(\sqrt{2})$ both means are 0.9937, and all 258 terminal sectors have $M/T \geq 0.9$. Empty sectors in the finite database occur almost entirely at small norm or in fields with shallow computational coverage.

3. Maeda-type heuristic

At prime level the Atkin–Lehner sign is the unavoidable local separator of Hecke orbits. After fixing that sign and removing CM and base-change forms, a generic Hecke polynomial is expected to have one irreducible factor of essentially full degree. Low-degree factors arising from geometric or congruence phenomena may persist, but their total dimension should be negligible.

The formulation is deliberately stronger than the statement that every fixed bounded Hecke degree has density zero among eigenforms. The latter allows a sector to decompose into, for example, $\sqrt{T}$ constituents of degree $\sqrt{T}$; then every fixed degree is negligible while $M/T \to 0$.

4. Relation with known results

The exact classical analogue is the prime-level, sign-by-sign conjecture of Lipnowski and Schaeffer. Kimball Martin records a closely related on-average Maeda-type conjecture in the level aspect. Dieulefait, Pacetti and Tsaknias identify local types and Atkin–Lehner data as natural separators of Galois orbits and propose Hilbert analogues in other aspects. Binder proves that bounded fields of rationality have density zero when eigenforms are counted with their field degrees. None of these results yields a constituent occupying a positive proportion of a fixed Hilbert sign sector.

Eventual nonemptiness of both signs is also separate. Prescribed inertial-type counting cannot distinguish the two unramified twists of a Steinberg representation, which have opposite Atkin–Lehner signs. A prime-level Hilbert trace estimate for the Atkin–Lehner involution would be needed.

5. The base-field involution

At an inert prime ideal, the nontrivial automorphism of $F/\mathbb{Q}$ preserves the level and acts on Hecke orbits. If the largest orbit were exchanged with a distinct orbit of the same degree, then $M/T \leq 1/2$. Consequently, Conjecture 1 predicts that the dominant orbit is eventually stable under the base-field involution, with that involution realized by an automorphism of its Hecke field. This is a genuine Hilbert-specific constraint and not merely a restatement of the classical conjecture.

6. Main obstacles

Two independent estimates are missing. First, if $V_{\mathfrak{p}}$ is the weight-two newspace, one needs

$$\operatorname{tr}(W_{\mathfrak{p}} \mid V_{\mathfrak{p}}) = o(\mathrm{N}\mathfrak{p})$$

with sufficient control of CM and base-change subspaces, in order to establish eventual nonemptiness and comparable dimensions of the two sign sectors. Second, one needs asymptotic simplicity of the remaining rational Hecke module. No trace formula, limit-multiplicity theorem, or rationality-field theorem currently produces a simple constituent of dimension $(1 - o(1))T_{\mathfrak{p},\varepsilon}$; the analogous assertion over $\mathbb{Q}$ remains conjectural.

References

[1] M. Lipnowski and G. J. Schaeffer, *Detecting large simple rational Hecke modules for $\Gamma_0(N)$ via congruences*, Int. Math. Res. Not. IMRN (2020), 6149–6168.

[2] K. Martin, *An on-average Maeda-type conjecture in the level aspect*, Proc. Amer. Math. Soc. **149** (2021), 1373–1386.

[3] L. Dieulefait, A. Pacetti and P. Tsaknias, *On the number of Galois orbits of newforms*, J. Eur. Math. Soc. **23** (2021), 2833–2860.

[4] J. Binder, *Fields of rationality of cusp forms*, Israel J. Math. **222** (2017), 973–1028.

[5] J. Weinstein, *Hilbert modular forms with prescribed ramification*, Int. Math. Res. Not. IMRN (2009), 1388–1420.

[6] L. Dembélé, *Compatibility between base change and Hecke orbits of Hilbert newforms*, arXiv:1711.05181.