MINIMALIST RANK STATISTICS IN ISOGENY-STRATIFIED FAMILIES OVER QUADRATIC FIELDS

Abstract. For a fixed quadratic field, we consider non-CM elliptic-curve isogeny classes that are not $\mathbb{Q}$-curves and impose an exact finite pattern of rational cyclic isogenies. We conjecture that every infinite such conductor-ordered stratum satisfies the minimalist rank law: asymptotic mass one half in each of ranks zero and one, and zero density in higher rank. We record the available numerical evidence and explain the additional root-number and Selmer-distribution inputs required beyond the usual minimalist conjecture.

1. The isogeny strata

Fix a quadratic field K, a finite set S of rational primes, and a vector $\eta = (\eta_\ell)_{\ell \in S} \in \{0,1\}^S$. For a K-isogeny class $\mathcal{C}$, let

$$I_\ell(\mathcal{C}) = \begin{cases} 1, & \text{if } \mathcal{C} \text{ admits a } K\text{-rational cyclic } \ell\text{-isogeny,} \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{G}_{K,S,\eta}(X)$ be the set of non-CM K-isogeny classes $\mathcal{C}$ such that

- $\mathcal{C}$ is not a $\mathbb{Q}$-curve isogeny class;
- $\mathrm{N}_{K/\mathbb{Q}} \mathfrak{N}_{\mathcal{C}} \le X$;
- $I_\ell(\mathcal{C}) = \eta_\ell$ for every $\ell \in S$.

Conjecture 1. *If $\#\mathcal{G}_{K,S,\eta}(X) \to \infty$, then*

$$\Pr_{\mathcal{C} \in \mathcal{G}_{K,S,\eta}(X)} \big(\operatorname{rank} \mathcal{C} = 0\big) \longrightarrow \frac{1}{2},$$
$$\Pr_{\mathcal{C} \in \mathcal{G}_{K,S,\eta}(X)} \big(\operatorname{rank} \mathcal{C} = 1\big) \longrightarrow \frac{1}{2},$$
$$\Pr_{\mathcal{C} \in \mathcal{G}_{K,S,\eta}(X)} \big(\operatorname{rank} \mathcal{C} \ge 2\big) \longrightarrow 0.$$

The exclusion of $\mathbb{Q}$-curves removes base-change and conjugate-isogeny mechanisms that can force atypical rank behavior. The negative conditions $I_\ell = 0$ are part of the statement; the conjecture is not merely about families possessing one specified isogeny.

Key words and phrases. elliptic curves over quadratic fields, cyclic isogenies, Mordell–Weil rank, minimalist conjecture, conductor ordering.

2. Numerical evidence

The available quadratic-field data contain 275404 non-CM, non-$\mathbb{Q}$-curve classes with known rank. The largest prime-isogeny strata are summarized in Table 1.

ℓ	total	rank 0	rank 1	rank ≥ 2
2	111910	50399	55980	5064
3	37328	18031	17835	1276
5	4562	2310	2062	162
7	1128	542	542	40

Table 1. Rank counts in classes admitting an ℓ-isogeny; rows with unknown rank are omitted from the displayed rank columns.

Exact small-prime signatures show the same qualitative pattern. Two examples are given in Table 2.

D_K	condition	total	rank 0	rank 1	rank ≥ 2
-11	2 but not $3, 5$	10386	4376	5213	772
-11	2 and 3	574	266	299	8
-3	2 but not $3, 5$	14574	6267	7546	746
-3	2 and 3	732	414	310	8

Table 2. Selected exact isogeny signatures. A small number of unknown ranks accounts for occasional discrepancies between row totals and displayed rank counts.

No rank at least three occurs in these four large exact-signature samples. On the other hand, the exact stratum having no isogeny of degree $2, 3, 5,$ or 7 still has a rank-at-least-two proportion of roughly 16.5%–18.9% at the largest available cutoffs. Thus the finite data support concentration in ranks zero and one, but also show that convergence can be very slow and highly dependent on the signature.

3. Heuristic motivation

A finite isogeny signature imposes finitely many conditions on residual Galois representations and selects rational points on products of modular curves. It should not, by itself, alter the orthogonal symmetry type underlying the minimalist model. Isogeny-Selmer groups can nevertheless be substantially larger in these families; the conjecture predicts that most of this excess is absorbed by Tate–Shafarevich groups rather than Mordell–Weil rank.

The strongest part of Conjecture 1 is not the density-zero assertion for rank at least two but the universal equality of the two remaining masses.

Over a number field, parity in a quadratic-twist family can have a nonzero local disparity. A more flexible formulation would replace 1/2 by the odd-root-number density $\theta_{K,S,\eta}$ and predict

$$\Pr(\text{rank} = 1) \to \theta_{K,S,\eta}, \qquad \Pr(\text{rank} = 0) \to 1 - \theta_{K,S,\eta}.$$

Conjecture 1 includes the additional assertion that every admissible exact isogeny stratum has $\theta_{K,S,\eta} = 1/2$.

4. Relation with known results

The Bhargava–Kane–Lenstra–Poonen–Rains model predicts the minimalist law in the full height-ordered family over a fixed global field. Work on elliptic curves with prescribed level structure provides conditional upper bounds for average analytic rank in certain genus-zero moduli families. For individual twist orbits, deep Selmer-distribution theorems prove bounded average rank and, under hypotheses, density-one rank at most one. Curves with a 3-isogeny over a number field also satisfy strong average-rank and positive-proportion results. None of these theorems establishes the exact conductor-ordered Mordell–Weil distribution in every finite isogeny-signature stratum.

5. Main obstacles

A proof needs four ingredients in the same family and ordering:

(1) an asymptotic count of the exact isogeny stratum by conductor norm;
(2) equidistribution of the two global root numbers in that stratum;
(3) minimal vanishing conditional on the root number, so that rank at least two has density zero;
(4) enough uniformity to combine the various modular-curve components while enforcing all negative isogeny conditions and the non-CM, non-$\mathbb{Q}$-curve exclusions.

Even when a high-genus modular curve has only finitely many K-points, each surviving geometric point has infinitely many quadratic twists. Consequently, some strata reduce to finite unions of twist families, where local parity disparity must be analyzed rather than assumed away.

References

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