TWO-TORSION MOMENTS IN LOCALLY CONDITIONED S_5-QUINTIC FIELDS

Abstract. We formulate conjectural first moments for the ordinary and narrow class-group 2-torsion of discriminant-ordered S_5-quintic fields. The fields are stratified simultaneously by signature, squarefreeness of the discriminant, and the set of common index divisors. The proposed constants are the usual fixed-signature Cohen–Lenstra–Martinet–Malle and Dummit–Voight moments, and the conjecture asserts that the additional local restrictions do not change them.

1. The conditioned families

Let $K/\mathbb{Q}$ be a quintic field whose Galois closure has group S_5. Write

$$(r_1, r_2) = (5 - 2r_2, r_2), \qquad r_2 \in \{0, 1, 2\},$$

and let

$$u = r_1 + r_2 - 1 = 4 - r_2$$

be the unit rank. The common index of K is

$$i(K) = \gcd_{\mathbb{Q}(\alpha)=K} [\mathcal{O}_K : \mathbb{Z}[\alpha]],$$

and we write

$$I(K) = \{p : p \mid i(K)\}.$$

For quintic fields one has $I(K) \subseteq \{2, 3\}$.

Let $\eta \in \{\mathrm{sf}, \mathrm{nsf}\}$ and $T \subseteq \{2, 3\}$. We denote by $\mathcal{F}_{r_2,\eta,T}(X)$ the set of $\mathbb{Q}$-isomorphism classes of such fields with

$$|D_K| \leq X, \qquad I(K) = T,$$

and with D_K squarefree precisely when $\eta = \mathrm{sf}$.

Conjecture 1. *Whenever $\#\mathcal{F}_{r_2,\eta,T}(X) \to \infty$, one has*

$$\lim_{X\to\infty} \frac{1}{\#\mathcal{F}_{r_2,\eta,T}(X)} \sum_{K\in\mathcal{F}_{r_2,\eta,T}(X)} |\operatorname{Cl}(K)[2]| = 1 + 2^{-u},$$

$$\lim_{X\to\infty} \frac{1}{\#\mathcal{F}_{r_2,\eta,T}(X)} \sum_{K\in\mathcal{F}_{r_2,\eta,T}(X)} |\operatorname{Cl}^{+}(K)[2]| = 1 + 2^{-r_2}.$$

The limits are independent of η and T.

The predicted values are displayed in Table 1.

Key words and phrases. quintic fields, class groups, narrow class groups, 2-torsion, common index divisors, squarefree discriminants.

r_2	signature	$\mathbb{E}\lvert \mathrm{Cl}(K)[2]\rvert$	$\mathbb{E}\lvert \mathrm{Cl}^+(K)[2]\rvert$
0	$(5,0)$	$17/16$	2
1	$(3,1)$	$9/8$	$3/2$
2	$(1,2)$	$5/4$	$5/4$

TABLE 1. The conjectural fixed-signature moments.

2. Numerical evidence

The available discriminant-complete windows contain 885295 fields: 162022 with $r_2 = 0$, 225089 with $r_2 = 1$, and 498184 with $r_2 = 2$. In the largest strata the empirical moments increase with the discriminant and move toward the predicted constants. Representative lower-shell and upper-shell values are given in Table 2.

r_2	η	moment	lower shell	upper shell	target
0	sf	$\lvert \mathrm{Cl}[2]\rvert$	1.005175	1.015640	1.0625
0	sf	$\lvert \mathrm{Cl}^+[2]\rvert$	1.664683	1.767668	2
1	sf	$\lvert \mathrm{Cl}[2]\rvert$	1.015390	1.040433	1.125
1	sf	$\lvert \mathrm{Cl}^+[2]\rvert$	1.271209	1.326719	1.5
2	sf	$\lvert \mathrm{Cl}[2]\rvert$	1.088132	1.129561	1.25
2	nsf	$\lvert \mathrm{Cl}[2]\rvert$	1.044010	1.088423	1.25

TABLE 2. Selected observations in the $T = \varnothing$ strata. For $r_2 = 2$, ordinary and narrow class groups agree.

Similar movement is visible after imposing $I(K) = \{2\}$. The strata with $I(K) = \{3\}$ or $\{2,3\}$ are much smaller, so the current data do not provide a meaningful asymptotic test there. The persistent deficit below the conjectural constants is compatible with slow convergence and with the sensitivity of an unbounded moment to rare fields of large 2-rank.

3. Heuristic background

The ordinary constant $1 + 2^{-u}$ is the modified Cohen–Lenstra–Martinet–Malle prediction for odd-degree S_n-fields, and the narrow constant $1 + 2^{-r_2}$ is the corresponding Dummit–Voight prediction. The conditions $I(K) = T$ are finite local restrictions at 2 and 3. Squarefreeness of the discriminant is an infinite collection of local ramification restrictions, but it is still a natural acceptable family in the quintic counting problem. Conjecture 1 asserts that neither type of conditioning changes the global 2-primary moment once the signature is fixed.

Class field theory gives a useful reformulation. The quantity $\lvert \mathrm{Cl}(K)[2]\rvert - 1$ counts nontrivial quadratic extensions of K unramified at all finite places and split at every real place, whereas $\lvert \mathrm{Cl}^+(K)[2]\rvert - 1$ counts quadratic extensions unramified at all finite places without the archimedean splitting requirement. A proof therefore requires asymptotic counts of such decorated towers over locally specified S_5-quintic fields.

4. Relation with known results

Bhargava's quintic parametrization gives linear discriminant-aspect counts and allows suitable local specifications; the geometric sieve also treats squarefree discriminants. In degree three, Bhargava and Varma prove analogous locally conditioned 2-torsion averages. Ho, Shankar and Varma obtain the expected bounds, and conditional equalities, in binary-form-parametrized odd-degree families. These results do not count every discriminant-ordered S_5-quintic field once, and no theorem presently gives even the unconditioned moments in Conjecture 1.

5. Main obstacles

A proof would require, in every fixed stratum,

$$\sum_{K\in\mathcal{F}_{r_2,\eta,T}(X)} (|\operatorname{Cl}(K)[2]|-1) \sim 2^{-u}\#\mathcal{F}_{r_2,\eta,T}(X),$$
$$\sum_{K\in\mathcal{F}_{r_2,\eta,T}(X)} (|\operatorname{Cl}^+(K)[2]|-1) \sim 2^{-r_2}\#\mathcal{F}_{r_2,\eta,T}(X).$$

The denominator is accessible through quintic-field counting, but no available parametrization counts the required unramified quadratic towers. One must also control the high-2-rank tail: convergence of fixed-rank frequencies alone does not imply convergence of the moment 2^{rk_2}.

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