INDEPENDENCE OF ROOT NUMBERS AT COMMON BIANCHI LEVEL

Abstract. Fix an imaginary quadratic field K. We pair genuine rational non-CM Bianchi newforms with exact rational-source base-change newforms having the same level ideal, and weight all ordered pairs equally. We conjecture that the two global root numbers become asymptotically independent. The statement admits an exact reformulation as the vanishing of a level-weighted covariance.

1. The common-level pair family

Fix an imaginary quadratic field K. Let $\mathcal{G}_K(X)$ denote the genuine, rational, non-CM Bianchi newform orbits of parallel weight two and trivial central character with level norm at most X. Let $\mathcal{B}_K(X)$ denote the exact base changes of rational classical weight-two newforms, again counted once as Bianchi orbits. Define

$$\mathcal{P}_K(X) = \{(F, G) \in \mathcal{G}_K(X) \times \mathcal{B}_K(X) : \mathfrak{N}_F = \mathfrak{N}_G\}.$$

Every ordered pair is assigned equal weight. For $\varepsilon, \delta \in \{\pm 1\}$, put

$$u_\varepsilon(X) = \Pr_{\mathcal{P}_K(X)}\big(w(F) = \varepsilon\big),$$
$$v_\delta(X) = \Pr_{\mathcal{P}_K(X)}\big(w(G) = \delta\big),$$
$$b_{\varepsilon,\delta}(X) = \Pr_{\mathcal{P}_K(X)}\big(w(F) = \varepsilon,\ w(G) = \delta\big).$$

Conjecture 1.1. *Assume that $\#\mathcal{P}_K(X) \to \infty$ and that every marginal sign probability has positive lower limit. Then, for every $\varepsilon, \delta \in \{\pm 1\}$,*

$$\lim_{X \to \infty} \frac{b_{\varepsilon,\delta}(X)}{u_\varepsilon(X) v_\delta(X)} = 1.$$

The exact common-level condition creates local arithmetic dependence. The conjecture asserts that, after this conditioning and after using the pair-weighted marginals, no residual correlation remains between the genuine and base-change origins.

2. Numerical evidence

For the five deepest fields, the positive-positive independence ratio is:

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D_K	number of ordered pairs	full range	norm $> 10^4$
-3	1,848	0.9037	0.9000
-4	1,012	0.9579	0.9506
-7	1,130	1.0004	1.0064
-8	1,204	0.9842	0.9885
-11	1,388	1.0061	1.0226

Table 1. Values of $b_{+,+}/(u_+v_+)$ in the current data.

Four fields are already close to independence, while $D_K = -3$ exhibits a persistent finite-range negative correlation.

3. An exact covariance reformulation

For an exact level ideal $\mathfrak{n}$, let $A_\varepsilon(\mathfrak{n})$ count genuine rational non-CM forms of sign ε, and let $B_\delta(\mathfrak{n})$ count exact rational-source base-change orbits of sign δ. Write

$$A = A_+ + A_-, \qquad \Delta A = A_+ - A_-,$$
$$B = B_+ + B_-, \qquad \Delta B = B_+ - B_-.$$

Set

$$T_X = \sum_{\mathrm{N}\mathfrak{n} \le X} A(\mathfrak{n})B(\mathfrak{n})$$

and

$$\mu_F = \frac{1}{T_X} \sum_{\mathrm{N}\mathfrak{n} \le X} \Delta A(\mathfrak{n})B(\mathfrak{n}),$$
$$\mu_G = \frac{1}{T_X} \sum_{\mathrm{N}\mathfrak{n} \le X} A(\mathfrak{n})\Delta B(\mathfrak{n}),$$
$$\mu_{FG} = \frac{1}{T_X} \sum_{\mathrm{N}\mathfrak{n} \le X} \Delta A(\mathfrak{n})\Delta B(\mathfrak{n}).$$

A direct calculation gives, for $\varepsilon, \delta \in \{\pm 1\}$,

$$b_{\varepsilon,\delta}(X) - u_\varepsilon(X)v_\delta(X) = \frac{\varepsilon\delta}{4}(\mu_{FG} - \mu_F\mu_G).$$

Since the marginal probabilities have positive lower limits, Conjecture 1.1 is equivalent to

$$\mu_{FG} - \mu_F\mu_G \longrightarrow 0.$$

Equivalently, with

$$q_X(\mathfrak{n}) = \frac{A(\mathfrak{n})B(\mathfrak{n})}{T_X}, \qquad \alpha(\mathfrak{n}) = \frac{\Delta A(\mathfrak{n})}{A(\mathfrak{n})}, \qquad \beta(\mathfrak{n}) = \frac{\Delta B(\mathfrak{n})}{B(\mathfrak{n})},$$

the conjecture is precisely

$$\mathrm{Cov}_{q_X}(\alpha, \beta) \longrightarrow 0.$$

4. Arithmetic obstruction and open problem

For a rational weight-two newform g of conductor N with $(N, D_K) = 1$, Artin formalism gives

$$\mathfrak{N}_{\mathrm{BC}_{K/\mathbf{Q}}(g)} = N\mathcal{O}_K, \qquad w(\mathrm{BC}_{K/\mathbf{Q}}(g)) = -\chi_K(N).$$

Thus the base-change sign is a deterministic quadratic character of the common level on the clean source-conductor stratum. Proving independence therefore requires decorrelation of the levelwise genuine-form sign bias from this fixed character. Primes dividing D_K introduce additional conductor-drop and epsilon-factor strata.

The growth and marginal assumptions alone do not imply the covariance estimate. An abstract level array can have balanced marginals but perfect same-sign correlation. The missing input is an arithmetic, product-multiplicity-weighted twisted first-moment estimate for rational genuine Bianchi orbits at the exact ideals supporting rational-source base change. Available trace formulas count full newspaces, usually with vector-space or spectral multiplicity, and do not isolate the rational degree-one slice.

There is also a formulation issue: in the standard literature, "genuine" excludes twists of base change, whereas the database condition "not base change and not CM" is broader. These two conventions define different pair families and should be distinguished explicitly in any final version of the conjecture.

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