ROOT NUMBERS IN EXCEPTIONAL-IMAGE FAMILIES OF GENUS-TWO JACOBIANS

Abstract. Fix an odd prime ℓ and prescribe whether ℓ divides the conductor. Among typical genus-two curves whose mod-ℓ representation is nonsurjective and whose Jacobian has no rational ℓ-torsion, we conjecture root-number equidistribution when curves are ordered by absolute minimal discriminant. The exclusions isolate the residual-image condition from the most immediate sources of finite-range sign bias.

1. The exceptional-image strata

Let ℓ be an odd prime and let $e \in \{0,1\}$. For $X \geq 1$, define

$$\mathcal{F}_{\ell,e}(X) = \left\{ C/\mathbf{Q} \;\middle|\; \begin{array}{l} C \text{ is a smooth projective curve of genus 2,} \\ |\Delta_{\min}(C)| \leq X, \qquad \mathrm{End}_{\overline{\mathbf{Q}}}(J_C) = \mathbf{Z}, \\ \operatorname{im} \bar{\rho}_{J_C,\ell} e \, \mathrm{GSp}_4(\mathbf{F}_\ell), \qquad J_C(\mathbf{Q})[\ell] = 0, \\ \mathbf{1}_{\ell \mid N_{J_C}} = e \end{array} \right\}.$$

Curves are counted up to $\mathbf{Q}$-isomorphism.

Conjecture 1.1. *If $|\mathcal{F}_{\ell,e}(X)| \to \infty$, then*

$$\lim_{X\to\infty} \frac{1}{|\mathcal{F}_{\ell,e}(X)|} \sum_{C \in \mathcal{F}_{\ell,e}(X)} w(J_C) = 0.$$

The conjecture fixes the reduction status at ℓ and removes rational ℓ-torsion. These conditions are intended to separate the exceptional residual-image constraint from two direct sources of local root-number bias.

2. Numerical evidence

The observed sign counts are:

Key words and phrases. genus two, root numbers, residual Galois images, exceptional primes, equidistribution.

ℓ	reduction at ℓ	$w = +1$	$w = -1$
3	good	247	259
3	bad	258	303
5	good	40	53
5	bad	30	36
7	good	8	7
7	bad	2	4

Table 1. Root-number counts after excluding rational ℓ-torsion.

Retaining rational ℓ-torsion creates visible finite-range positive-sign biases for $\ell = 5$ and 7, which motivates its exclusion from the conjectural family.

3. Why existing methods do not prove the conjecture

Explicit local formulas express $w(J_C)$ as a product of local signs, but do not imply cancellation after imposing the thin global condition

$$\operatorname{im} \bar{\rho}_{J_C,\ell} e \, \mathrm{GSp}_4(\mathbf{F}_\ell).$$

Known equidistribution results for twisted Fermat Jacobians concern families with extra geometric endomorphisms and different height parameters. Automorphic root-number equidistribution theorems use varying-weight or full spectral families and do not isolate these fixed-weight geometric strata.

A quadratic-twist pairing is insufficient. Twisting often preserves the geometric endomorphism ring and the exceptional residual image, but it changes the minimal discriminant nonuniformly and need not reverse the root number. Bisatt's lawful genus-two examples show that an entire quadratic-twist family can have constant root number. Conversely, an infinite constant-sign subfamily would not disprove Conjecture 1.1 without a positive-density statement inside the full exceptional-image stratum.

4. The required signed count

A proof would require

$$S_{\ell,e}(X) := \sum_{C \in \mathcal{F}_{\ell,e}(X)} w(J_C) = o\big(|\mathcal{F}_{\ell,e}(X)|\big)$$

for every fixed odd ℓ and $e \in \{0,1\}$. Thus one needs both an asymptotic count of typical genus-two curves with nonsurjective mod-ℓ image, no rational ℓ-torsion, and prescribed reduction status, and a power-saving or otherwise sufficient estimate for the same count weighted by the global root number. Existing thin-set, Hilbert-irreducibility, open-image, Selmer-parity, and local-root-number results do not provide this discriminant-ordered signed asymptotic. Infinitude of the denominator alone gives no cancellation mechanism.

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