SIMULTANEOUS EXCEPTIONAL PRIMES FOR TYPICAL GENUS-TWO JACOBIANS

ABSTRACT. For a genus-two curve $C/\mathbf{Q}$ with $\mathrm{End}_{\overline{\mathbf{Q}}}(J_C) = \mathbf{Z}$, let $\mathrm{Exc}(C)$ be the set of primes at which the residual Galois representation is not surjective. We conjecture the uniform cardinality bound $|\mathrm{Exc}(C)| \leq 3$. This bound is attained in the current data, while no example with four exceptional primes is known.

1. Statement

For a smooth projective genus-two curve $C/\mathbf{Q}$, define

$$\mathrm{Exc}(C) = \{\ell : \mathrm{im}\,\bar{\rho}_{J_C,\ell} \neq \mathrm{GSp}_4(\mathbf{F}_\ell)\}.$$

The set records distinct primes, not distinct maximal subgroups or image labels.

Conjecture 1.1. *If*

$$\mathrm{End}_{\overline{\mathbf{Q}}}(J_C) = \mathbf{Z},$$

then

$$|\mathrm{Exc}(C)| \leq 3.$$

Serre's theorem implies that $\mathrm{Exc}(C)$ is finite for each fixed typical Jacobian. The conjecture asks for a uniform bound on the number of simultaneous failures, rather than a uniform bound on the size of each exceptional prime.

2. Numerical evidence and sharpness

Among the 63,107 typical LMFDB curves, the distribution of the number of exceptional primes is:

$\lvert\mathrm{Exc}(C)\rvert$	0	1	2	3
number of curves	19,586	40,192	3,306	23

TABLE 1. Number of simultaneous exceptional primes in the inspected data.

The three-prime sets are $\{2, 3, 5\}$ for 20 curves, $\{2, 3, 7\}$ for two curves, and $\{2, 3, 13\}$ for one curve. In particular, the curve with LMFDB label `1116.a.214272.1` realizes the set $\{2, 3, 13\}$, so the proposed bound is sharp if true.

Key words and phrases. genus two, Jacobians, residual Galois representations, exceptional primes, uniformity.

A larger published computation of 1,743,737 typical curves likewise found at most three exceptional primes: 199,183 curves had none, 1,394,671 had one, 148,606 had two, and 1,277 had three.

3. What is known

Open-image and maximal-subgroup results analyze exceptional primes one at a time. Serre proves only curvewise finiteness. Banwait–Brumer–Kim–Klagsbrun–Mayle–Srinivasan–Vogt give fixed-curve algorithms and a GRH-conditional bound for the product of exceptional primes in terms of the conductor. Lombardo gives a large-prime cutoff depending on the height and arithmetic data of the individual surface. None of these results couples exceptional behavior at several distinct primes.

Rational ℓ-torsion fixes a nonzero vector in $J_C[\ell]$, and a rational ℓ-isogeny stabilizes a proper subgroup; either forces $\ell \in \mathrm{Exc}(C)$. These observations suggest possible counterexample constructions, but no typical Jacobian with four distinct primes forced in this way is known. Howe's order-70 construction does not qualify because its Jacobian is geometrically isogenous to a product.

4. The mixed-level problem

A proof of Conjecture 1.1 would require a genuinely cross-prime uniform theorem. For every four distinct primes ℓ_i and every choice of proper maximal subgroups

$$H_i < \mathrm{GSp}_4(\mathbf{F}_{\ell_i}),$$

one would need to show that every rational point on the associated mixed-level Siegel modular cover either represents a surface with extra geometric endomorphisms or does not arise from a genus-two Jacobian. Existing open-image, isogeny-bound, and maximal-subgroup methods do not control rational points on all such fiber products.

A disproof would require one explicit genus-two curve with geometric endomorphism ring $\mathbf{Z}$ and four rigorously certified nonsurjective residual images. No such example is presently known.

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