ELLIPTIC REALIZATION OF RATIONAL GENUINE BIANCHI NEWFORMS

Abstract. Fix an imaginary quadratic field K. Among genuine, non-CM, rational Bianchi newforms of parallel weight two and trivial central character, we conjecture that the forms realized by elliptic curves over K have density one when ordered by level norm. The complementary quaternionic-multiplication branch is infinite in some fields but is constrained to squarefull levels; proving density zero nevertheless requires new conductor-aspect counting results.

1. The realization problem

Fix an imaginary quadratic field K. Let $\mathcal{G}_K(X)$ be the set of Galois orbits of genuine, non-CM Bianchi newforms over K of parallel weight two, trivial central character, rational Hecke field, and level norm at most X. Each orbit is counted once. Define

$$\mathcal{E}_K(X) = \left\{ F \in \mathcal{G}_K(X) : \begin{array}{l} \text{there exists an elliptic curve } E/K \text{ such that} \\ L(E,s) = L(F,s) \end{array} \right\}.$$

Conjecture 1.1. *If $\#\mathcal{G}_K(X) \to \infty$, then*

$$\lim_{X\to\infty} \frac{\#\mathcal{E}_K(X)}{\#\mathcal{G}_K(X)} = 1.$$

Equivalently, under the standard elliptic/QM Eichler–Shimura dichotomy, the subfamily realizable only by quaternionic-multiplication abelian surfaces has density zero.

The dichotomy itself is conjectural in general. Both branches are allowed by the expected motivic correspondence, but the QM branch imposes additional endomorphisms and should lie on lower-dimensional Shimura-type loci.

2. Numerical evidence

Among 215,090 genuine non-CM rational records, 214,931 have an elliptic realization, 106 have no associated elliptic curve, and 53 remain unresolved. The deepest fixed fields with no unresolved rows give:

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D_K	elliptically realized / total
-3	40,968/40,980
-4	38,828/38,832
-7	34,804/34,804
-8	35,486/35,486
-11	29,430/29,430

Table 1. Elliptic realizations in the five deepest fixed fields.

The non-elliptic rows occur in small packets at only 22 field–level-norm combinations rather than as a visibly positive fraction of the data.

3. Known structural constraints

Taylor's conjectural Eichler–Shimura statement, in the form made explicit by Schembri, predicts that a rational weight-two Bianchi newform corresponds either to an elliptic curve E/K with $L(E,s) = L(F,s)$ or to a QM surface A/K with

$$L(A,s) = L(F,s)^2.$$

Schembri constructs genuine division-QM examples, so universal elliptic realization is false.

If a rational non-CM form has a division-QM realization A, results of Guitart–Masdeu imply

$$\mathrm{cond}(A) = \mathfrak{N}_F^2 \quad \text{and} \quad v_{\mathfrak{p}}(\mathrm{cond}(A)) \geq 4 \quad \text{at every bad prime } \mathfrak{p}.$$

Consequently

$$v_{\mathfrak{p}}(\mathfrak{N}_F) \geq 2,$$

so every such form lies at a squarefull level. There are only $O_K(X^{1/2})$ squarefull ideals of norm at most X, but this alone does not control the number of rational newforms at each level.

4. An infinite exceptional subfamily

The QM-only subfamily is not finite in at least two fixed fields.

Proposition 4.1. *For $K = \mathbf{Q}(i)$ and $K = \mathbf{Q}(\sqrt{-3})$, there are infinitely many genuine, non-CM, rational Bianchi newforms with trivial central character that have a division-QM realization and no elliptic realization.*

Proof sketch. Choose one of Schembri's genuine division-QM forms π over K and let χ vary over quadratic Hecke characters of K. Then $\pi \otimes \chi$ remains rational, of weight two, and of trivial central character. Genuineness and the absence of CM are preserved under quadratic twisting. The corresponding twisted QM surface still realizes $\pi \otimes \chi$.

If $\pi \otimes \chi$ admitted an elliptic realization, twisting back by χ would give an elliptic realization of π, contrary to the choice of the seed. Distinct characters give distinct forms, since a collision would give π a nontrivial quadratic self-twist. For conductors coprime to the seed level,

$$\mathrm{N}\mathfrak{N}_{\pi\otimes\chi} = \mathrm{N}\mathfrak{N}_{\pi}\,\mathrm{N}\mathfrak{f}(\chi)^2.$$

The quadratic-extension asymptotic therefore gives a lower bound of order $X^{1/2}$ for this twist family. □

This does not contradict Conjecture 1.1: the total number of genuine rational forms may grow faster than $X^{1/2}$.

5. The remaining counting problem

Let

$$Q_K(X) = \#(\mathcal{G}_K(X) \setminus \mathcal{E}_K(X)).$$

The conjecture asks for $Q_K(X) = o(\#\mathcal{G}_K(X))$. Three major inputs are missing: the elliptic/QM dichotomy is not known for all rational Bianchi forms; there is no conductor-aspect upper bound for twist-minimal QM systems or for the multiplicity of rational forms at squarefull levels; and there is no suitable lower bound or asymptotic for $\#\mathcal{G}_K(X)$. In the two fields above, density zero would in particular require growth of the denominator faster than the established $X^{1/2}$ exceptional twist family.

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