A UNIFORM SURJECTIVITY BOUND FOR TYPICAL GENUS-TWO JACOBIANS

Abstract. Let $C/\mathbf{Q}$ be a genus-two curve whose Jacobian has geometric endomorphism ring $\mathbf{Z}$. We conjecture that its residual Galois representation is surjective onto $\mathrm{GSp}_4(\mathbf{F}_\ell)$ for every prime $\ell > 31$. The value 31 is compatible with the current large-scale computations and is known to be a necessary lower boundary, but even the existence of an absolute uniform bound remains open.

1. Statement

Let $C/\mathbf{Q}$ be a smooth projective curve of genus two and let

$$\bar\rho_{J_C,\ell} : G_{\mathbf{Q}} \longrightarrow \mathrm{GSp}_4(\mathbf{F}_\ell)$$

be the representation on $J_C[\ell]$, defined using the Weil pairing.

Conjecture 1.1. *If*

$$\mathrm{End}_{\overline{\mathbf{Q}}}(J_C) = \mathbf{Z},$$

then

$$\mathrm{im}\,\bar\rho_{J_C,\ell} = \mathrm{GSp}_4(\mathbf{F}_\ell) \qquad \textit{for every prime } \ell > 31.$$

The endomorphism hypothesis is the usual “typical” condition for an abelian surface. Serre’s open-image theorem gives the same conclusion for all sufficiently large primes, but with a threshold depending on the individual Jacobian. Conjecture 1.1 replaces this curve-dependent threshold by the universal value 31.

2. Computational evidence

Among the 63,107 LMFDB curves in the typical stratum, the exceptional primes and their frequencies are:

ℓ	2	3	5	7	11	13	17	29
frequency	42,230	3,558	939	120	8	15	2	1

Table 1. Exceptional-prime frequencies in the inspected LMFDB stratum.

The largest exceptional prime in this sample is 29. A larger published computation found an exceptional prime 31 and no exceptional prime above

Key words and phrases. genus two, Jacobians, Galois representations, exceptional primes, uniformity, GSp(4).

31 among 1,743,737 curves. A certified typical Jacobian with a rational 31-isogeny shows that no uniform cutoff below 31 can hold.

3. Known results

For each fixed principally polarized abelian surface $A/\mathbf{Q}$ with $\mathrm{End}_{\overline{\mathbf{Q}}}(A) = \mathbf{Z}$, Serre proves surjectivity for all sufficiently large ℓ. Lombardo gives explicit bounds depending on the Faltings height and on arithmetic and reduction data. Banwait–Brumer–Kim–Klagsbrun–Mayle–Srinivasan–Vogt give algorithms for computing the exceptional set of an individual typical Jacobian and a conditional conductor-dependent bound.

These results do not imply a uniform constant. The multiplier of the residual representation is the surjective mod-ℓ cyclotomic character, but this does not force the symplectic part to be full. Proper maximal-image possibilities include reducible images, stabilizers of a line or a Lagrangian plane, imprimitive decompositions, and special subgroup types. Present methods eliminate them only with constants depending on the particular surface.

4. The uniformity gap

A proof of Conjecture 1.1 must uniformly exclude every proper maximal subgroup of $\mathrm{GSp}_4(\mathbf{F}_\ell)$ for every typical genus-two Jacobian and every prime $\ell \geq 37$. In particular, no theorem uniformly rules out rational ℓ-isogenies, imprimitive images, or the remaining special-image cases independently of height, conductor, and reduction data.

The distinction in quantifiers is essential:

$$\forall J\ \exists B(J) \quad \text{is known, whereas} \quad \exists B\ \forall J \quad \text{is open.}$$

The conjecture proposes the sharp value $B = 31$. Current computations provide substantial evidence but do not settle this universal assertion.

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