# ROOT NUMBERS OF RATIONAL QUADRATIC BASE-CHANGE BIANCHI NEWFORMS

Abstract. Fix an imaginary quadratic field $K$. We consider non-CM rational Bianchi newforms that arise by exact cyclic base change from rational classical newforms of weight two. Ordered by the norm of their Bianchi level and counted once as Bianchi Galois orbits, we conjecture that their global root numbers are equidistributed. Artin formalism reduces the problem to a fixed quadratic-character cancellation problem for elliptic curves ordered essentially by conductor.

## 1. The base-change family

Fix an imaginary quadratic field $K$ with discriminant character $\chi_K$. For $X \geq 1$, let

$$\mathcal{B}_K(X) = \left\{ F \;\middle|\; \begin{array}{l} F = \mathrm{BC}_{K/\mathbf{Q}}(f), \quad f \text{ is a classical newform of weight 2,} \\ \mathbf{Q}_f = \mathbf{Q}, \quad F \text{ is cuspidal and non-CM,} \quad \mathrm{N}\mathfrak{N}_F \leq X \end{array} \right\}.$$

Only exact cyclic base changes are included, and each resulting Bianchi Galois orbit is counted once.

**Conjecture 1.1.** *If $\#\mathcal{B}_K(X) \to \infty$, then for each $\varepsilon \in \{\pm 1\}$,*

$$\lim_{X \to \infty} \frac{\#\{F \in \mathcal{B}_K(X) : w(F) = \varepsilon\}}{\#\mathcal{B}_K(X)} = \frac{1}{2}.$$

## 2. Numerical evidence

For the exact rational-source base-change stratum, the deepest fields in the current data give:

| $D_K$ | number of forms | proportion with $w = +1$ |
|---|---|---|
| $-3$ | 944 | 0.4883 |
| $-4$ | 897 | 0.4928 |
| $-7$ | 509 | 0.4637 |
| $-8$ | 792 | 0.4874 |
| $-11$ | 496 | 0.5242 |
| $-19$ | 185 | 0.4865 |

Table 1. Observed root-number proportions in fixed imaginary quadratic fields.

*Key words and phrases.* Bianchi modular forms, quadratic base change, root numbers, elliptic curves, equidistribution.

The finite-range biases generally decrease as the level cutoff increases; for $D_K = -3$, the positive proportion falls from 0.8333 below norm $10^3$ to 0.5476 below $10^4$ and to 0.4883 in the full recorded range.

## 3. Reduction to a quadratic-character average

Let $E/\mathbf{Q}$ be the elliptic curve attached to the rational newform $f$, and write $E^{D_K}$ for its quadratic twist. Artin formalism gives

$$L(E/K,s) = L(E/\mathbf{Q},s)L(E^{D_K}/\mathbf{Q},s), \qquad w(\mathrm{BC}_{K/\mathbf{Q}}(f)) = w(E)w(E^{D_K}).$$

If $(N_E, D_K) = 1$, the quadratic-twist formula yields

$$w(E^{D_K}) = w(E)\chi_K(-N_E), \qquad w(\mathrm{BC}_{K/\mathbf{Q}}(f)) = \chi_K(-N_E).$$

Non-CM cyclic base change has exactly the two sources $f$ and $f \otimes \chi_K$. A fixed point would be a $\chi_K$-self-twist and hence a CM or dihedral form, which is excluded. Thus passing from sources to once-counted Bianchi orbits divides both the signed and unsigned counts by two.

At primes not dividing $D_K$, base extension contributes the square of the source conductor exponent. Hence, for fixed $K$,

$$\mathrm{N}\mathfrak{N}_{E/K} = N_E^2 c_K(E),$$

where $c_K(E)$ depends only on the finitely many local types at primes dividing $D_K$ and ranges over a finite set. Consequently Conjecture 1.1 is equivalent to cancellation in the average of $w(E)w(E^{D_K})$ over the corresponding conductor strata; on the coprime stratum, it is the cancellation of $\chi_K(-N_E)$.

## 4. Known results and the remaining gap

Cyclic base change constructs the family, while standard root-number formulas provide the preceding reduction. Existing results on quadratic twists fix $E$ and vary the twisting character, whereas the present problem fixes $K$ and varies $E$. Trace formulas for complete newspaces average over all coefficient fields and weight Galois orbits by their degrees; they do not isolate the thin rational-orbit subfamily.

For every fixed nontrivial $\chi_K$, one would need

$$\sum_E w(E)w(E^{D_K}) = o(\#\{E\}),$$

with $E$ ordered through the Bianchi conductor condition. On the coprime stratum this becomes

$$\sum_E \chi_K(-N_E) = o(\#\{E\}),$$

together with analogous locally corrected estimates at primes dividing $D_K$. No available theorem gives the required conductor-ordered asymptotic for all rational isogeny classes, let alone its distribution between the two quadratic-character classes.

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