A RANK-DEPENDENT SECONDARY BIAS IN THE SIGN OF THE MINIMAL DISCRIMINANT

Abstract. We study singleton non-CM isogeny classes of elliptic curves over $\mathbf{Q}$ with prime conductor. For each fixed rank $0 \le r \le 3$, we conjecture a two-term asymptotic for the proportion having negative minimal discriminant. The leading constant is the classical $\sqrt{3}$ archimedean ratio, while the secondary constants exhibit a strict rank-dependent ordering suggested by the available data.

1. Statement of the conjecture

For $r \in \{0,1,2,3\}$ and $X \ge 1$, let $\mathcal{P}_r(X)$ be the set of $\mathbf{Q}$-isogeny classes $[E]$ such that

(i) the conductor is a prime $N_E = p \le X$;
(ii) the isogeny class contains a single $\mathbf{Q}$-isomorphism class;
(iii) E is non-CM and $\operatorname{rk} E(\mathbf{Q}) = r$.

The sign of the minimal discriminant is then unambiguous on the isogeny class. Let $\mathcal{P}_r^-(X)$ denote the subset for which $\Delta_{\min}(E) < 0$.

Conjecture 1.1. *Assume that $\#\mathcal{P}_r(X) \to \infty$ for each $0 \le r \le 3$. There exist real constants d_0, d_1, d_2, d_3 such that*

$$\frac{\#\mathcal{P}_r^-(X)}{\#\mathcal{P}_r(X)} = \frac{\sqrt{3}}{1+\sqrt{3}} + \frac{d_r}{\log X} + o\left(\frac{1}{\log X}\right),$$

and these constants satisfy

$$d_0 > d_1 > 0 > d_2 > d_3.$$

The main term is the archimedean mass predicted by the Brumer–McGuinness and Watkins model. The conjecture asserts that conditioning on exact Mordell–Weil rank introduces a stable secondary correction of order $1/\log X$.

2. Numerical evidence

At $X = 3 \cdot 10^8$, the observed proportions and rescaled deviations are as follows.

Key words and phrases. elliptic curves, prime conductor, minimal discriminant, rank, secondary asymptotics.

r	$\#\mathcal{P}_r^-(X)/\#\mathcal{P}_r(X)$	$\log X\left(\dfrac{\#\mathcal{P}_r^-(X)}{\#\mathcal{P}_r(X)} - \dfrac{\sqrt{3}}{1+\sqrt{3}}\right)$
0	0.663154	0.56956
1	0.637888	0.07639
2	0.598767	−0.68723
3	0.557980	−1.48336

TABLE 1. Rank-conditioned discriminant-sign statistics at $X = 3 \cdot 10^8$.

Here $\sqrt{3}/(1+\sqrt{3}) = 0.633975\ldots$. Between $3 \cdot 10^7$ and $3 \cdot 10^8$, the four rescaled quantities remain near 0.59, 0.07, −0.66, and −1.47, respectively. The inspected sample contains 731,713 prime-conductor representatives, with no missing discriminant signs and with algebraic rank agreeing with analytic rank throughout.

3. Relation to known counting results

Brumer and McGuinness first observed the overall negative-to-positive ratio near $\sqrt{3}$ in prime-conductor data, and Watkins rederived the corresponding sign-dependent volume heuristic. Shankar–Shankar–Wang prove, for certain large conductor-ordered families defined by local conditions, sign-separated leading asymptotics whose constants satisfy $\alpha_- = \sqrt{3}\,\alpha_+$. Their theorem does not cover the thin requirement that the conductor be prime, nor does it condition on exact rank or give a secondary term.

By modularity and the Mestre–Oesterlé prime-conductor theorem, a prime-conductor isogeny class contains a curve with prime absolute minimal discriminant. In the singleton case this is the unique curve, so the conjecture can equivalently be viewed as a statement about exact-rank prime-discriminant curves. This reduction does not provide the required distribution.

4. The unresolved analytic problem

Let $A_{r,+}(X)$ and $A_{r,-}(X)$ be the positive- and negative-discriminant counts. Proving Conjecture 1.1 requires a rank- and sign-separated two-term asymptotic for these counts, strong enough to identify both the main ratio and the limits

$$d_r = \lim_{X\to\infty} \log X\left(\frac{A_{r,-}(X)}{A_{r,+}(X)+A_{r,-}(X)} - \frac{\sqrt{3}}{1+\sqrt{3}}\right).$$

The passage to prime conductor already involves a thin global sieve. Exact-rank conditioning further introduces global quantities such as central L-values, Selmer groups, regulators, and Tate–Shafarevich groups. Present lattice-counting and average-Selmer methods do not control these conditions simultaneously, and no existing theorem gives even the required leading exact-rank asymptotics in this family. The numerical pattern is therefore evidence rather than a proof.

References

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