ROOT NUMBERS OF BINARY-TETRAHEDRAL ARTIN REPRESENTATIONS

Abstract. We consider faithful irreducible two-dimensional Artin representations of $G_{\mathbf{Q}}$ with image $\mathrm{SL}_2(\mathbf{F}_3)$, trivial determinant, and Frobenius–Schur indicator -1. Ordered by Artin conductor, we conjecture that this family is infinite and that its global root numbers are equidistributed. We also record a proof of infinitude by quadratic twisting and explain why this construction does not itself produce sign cancellation.

1. The family and the conjecture

Let $q(\rho)$ and $w(\rho) \in \{\pm 1\}$ denote the Artin conductor and global root number of a finite-image complex representation ρ of $G_{\mathbf{Q}}$. For $X \geq 1$, set

$$\begin{aligned}\mathcal{T}(X) = \{[\rho] : \rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{C}) &\text{ is continuous, faithful, and irreducible,}\\ &\operatorname{im}\rho \simeq \mathrm{SL}_2(\mathbf{F}_3), \quad \mathrm{FS}(\rho) = -1,\\ &\det\rho = 1, \quad q(\rho) \leq X\}.\end{aligned}$$

where representations are counted up to complex isomorphism. No restriction is imposed on the projective A_4-field or on the set of ramified primes.

Conjecture 1.1. *The cardinality of $\mathcal{T}(X)$ tends to infinity, and*

$$\lim_{X\to\infty} \frac{1}{\#\mathcal{T}(X)} \sum_{[\rho]\in\mathcal{T}(X)} w(\rho) = 0.$$

Binary-tetrahedral representations form the first non-dihedral, projectively exceptional symplectic family in dimension two. Tetrahedral automorphy makes their root numbers analogous to Atkin–Lehner signs, while variation of the projective A_4-field and of the local lifting data provides a plausible source of cancellation.

2. Numerical evidence

The recorded LMFDB data give the following cumulative sign counts.

Key words and phrases. Artin representations, binary tetrahedral group, root numbers, conductor aspect, equidistribution.

X	$w = +1$	$w = -1$
10^5	4	4
$2.5 \cdot 10^5$	12	18
$5 \cdot 10^5$	34	42
10^6	49	60
$1.7 \cdot 10^6$	56	73

Table 1. Cumulative root-number counts. At the last cutoff, 129 of the 133 recorded orbits have known sign.

Both signs occur repeatedly. The full target table contains 278 rows and shows no determinant variation or hidden duplication in the inspected fields.

3. An infinitude result

The infinitude clause of Conjecture 1.1 follows from a quadratic-twist construction.

Proposition 3.1. *There are infinitely many isomorphism classes of representations satisfying all the defining conditions of $\mathcal{T}(X)$.*

Proof. Caputo and Vinatier construct a tame Galois extension $N_0/\mathbf{Q}$ with group $G \simeq \mathrm{SL}_2(\mathbf{F}_3)$ and a faithful irreducible symplectic representation ρ_0 of degree two and root number -1. Let $q_0 = q(\rho_0)$.

For every prime $\ell \equiv 1 \pmod 4$ with $\ell \nmid q_0$, let χ_ℓ be the primitive quadratic character of conductor ℓ and put $\rho_\ell = \rho_0 \otimes \chi_\ell$. Since $G^{\mathrm{ab}} \simeq C_3$, the extension N_0 is linearly disjoint from $\mathbf{Q}(\sqrt{\ell})$. The joint image of (ρ_0, χ_ℓ) is therefore $G \times \{\pm 1\}$, and the map $(g, e) \mapsto eg$ has image G and kernel $\{(I, 1), (-I, -1)\}$. Hence $\operatorname{im} \rho_\ell \simeq G$.

Scalar quadratic twisting preserves irreducibility, the alternating form, and the determinant. At ℓ, inertia acts by the scalar $-I$, so the inertia-fixed space is zero and the local conductor exponent is two; at all other primes the conductor is unchanged. Thus

$$q(\rho_\ell) = q_0 \ell^2.$$

Distinct primes give distinct conductors. Dirichlet's theorem consequently yields

$$\#\mathcal{T}(X) \geq \#\left\{\ell \equiv 1 \pmod 4 :\ \ell \nmid q_0,\ \ell \leq \sqrt{X/q_0}\right\} \longrightarrow \infty.$$

□

This family does not prove the sign assertion. The local epsilon-factor computation used by Shankar–Södergren–Templier gives $w(\rho_0 \otimes \chi_\ell) = w(\rho_0) = -1$ for these coprime twists. Thus the conjectural cancellation must occur between different projective A_4-fields, different reduced-Schur lifting classes, or different wild local types, rather than inside the twists of a single representation.

4. Relation to existing results and remaining problem

Rubinstein-Salzedo conjectures equidistribution of a reduced Schur lifting invariant in a restricted tame, totally real family of A_4-fields. Dalal and Gerbelli-Gauthier prove root-number equidistribution in a varying-weight automorphic family, but their hypotheses do not cover a fixed finite-image conductor aspect. Even the required conductor-ordered count of the relevant A_4-fields with prescribed central lift and local conditions is not currently available.

Writing

$$S(X) = \sum_{[\rho]\in\mathcal{T}(X)} w(\rho),$$

the unresolved assertion is $S(X) = o(\#\mathcal{T}(X))$. A proof would require sufficiently uniform counting, ordered by the two-dimensional Artin conductor, for liftable A_4-fields with prescribed local conditions and prescribed lifting or root-number invariant. The finite computation is consistent with the conjecture but does not establish this signed asymptotic.

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