ROOT-NUMBER EQUIDISTRIBUTION FOR GENUINE RATIONAL BIANCHI NEWFORMS

Abstract. We conjecture that the two global root numbers occur with equal limiting frequency among genuine non-CM rational Bianchi newforms of parallel weight two and trivial central character over a fixed imaginary quadratic field, ordered by level norm. We describe the finite-level bias visible in current data and explain the obstruction created by constant-sign quadratic-twist families.

1. The family and the conjecture

Retain the notation $\mathcal{R}_K(X)$ and $\mathcal{L}_K(X)$ for rational non-CM Bianchi newforms and their twisted-base-change subfamily. Define the genuine family by

$$\mathcal{G}_K(X) = \mathcal{R}_K(X) \setminus \mathcal{L}_K(X).$$

For $F \in \mathcal{G}_K(X)$, let $w(F) \in \{\pm 1\}$ be the sign in the functional equation of its completed standard L-function.

Conjecture 1. *For every imaginary quadratic field K, if $\#\mathcal{G}_K(X) \to \infty$, then for each $\varepsilon \in \{\pm 1\}$,*

$$\lim_{X\to\infty} \frac{\#\{F \in \mathcal{G}_K(X) : w(F) = \varepsilon\}}{\#\mathcal{G}_K(X)} = \frac{1}{2}.$$

Equivalently, if

$$A_K(X) = \#\mathcal{G}_K(X), \qquad S_K(X) = \sum_{F\in\mathcal{G}_K(X)} w(F),$$

then Conjecture 1 is the assertion

$$S_K(X) = o(A_K(X)).$$

2. Numerical evidence

The available data contain 215090 genuine non-CM rational orbits, of which 98391 have sign $+1$ and 116699 have sign -1. The pooled proportion is not the conjectured statistic because the level coverage differs strongly from one field to another.

For the deepest fixed-field datasets, the terminal positive proportions are:

The deepest fields show an initial bias toward sign -1 that weakens as the norm cutoff grows. For example, the field of discriminant -3 moves from

Key words and phrases. Bianchi modular forms, root numbers, Atkin–Lehner signs, equidistribution.

Disc(K)	-3	-4	-7	-8	-11	-19	-23	-31
$\Pr(w=+1)$	.4470	.4459	.4514	.4573	.4729	.5005	.4712	.4725

Table 1. Positive root-number proportions at the largest available fixed-field cutoffs.

positive proportion 0.4280 at norm 10000 to 0.4470 at norm 149968. Both signs occur in every meaningfully populated fixed-field sample.

3. Heuristic motivation

The global root number is a product of local signs. Once CM forms and all base-change mechanisms are removed, no visible global symmetry forces one sign. This suggests cancellation in a sufficiently large fixed-field level family, in analogy with classical sign-equidistribution results.

The rationality and genuineness restrictions are global and very thin. A trace formula on the full newspace gives a weighted sum

$$\sum_F [\mathbb{Q}_F : \mathbb{Q}] w(F),$$

not the unweighted sum over rational genuine orbits required here. There is no known projector imposing simultaneously $\mathbb{Q}_F = \mathbb{Q}$, non-CM status and exclusion of every twisted-base-change orbit.

4. The quadratic-twist obstruction

There are rational genuine level-one forms over $K = \mathbb{Q}(\sqrt{-643})$. Let π be one such form. For a quadratic Hecke character χ/K, twisting preserves parallel weight 2, rational Hecke eigenvalues, trivial central character, non-CM status and genuineness. Since π is unramified at every finite place, the local twisting formula and the product formula give

$$w(\pi \otimes \chi) = w(\pi)$$

for every quadratic χ. Non-CM status rules out nontrivial quadratic self-twists, so this produces infinitely many distinct genuine rational forms with the same root number.

This does not disprove Conjecture 1: the constant-sign twist family may have density zero inside $\mathcal{G}_K(X)$. It does show that equidistribution cannot be proved by a naive sign-reversing quadratic-twist pairing.

5. Relation with known results

Classical trace-formula results prove root-number equidistribution in full newspaces, and recent automorphic results treat self-dual families in other aspects. These theorems do not isolate the rational genuine Bianchi locus. Existing structural work describes the genuine/non-genuine decomposition and provides examples of rational genuine forms, but no signed counting theorem in the level aspect is known.

6. Open difficulties

For each fixed K, one needs both a sufficiently regular lower bound for $A_K(X)$ and the signed estimate

$$\sum_{F \in \mathcal{G}_K(X)} w(F) = o\big(A_K(X)\big).$$

It is unknown whether the explicit constant-sign quadratic-twist families are negligible, have positive density, or are balanced by other twist classes. A proof would therefore require a counting theory for genuine rational Bianchi twist classes that is substantially finer than current dimension formulae.

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