GENUINE RATIONAL BIANCHI NEWFORMS IN THE LEVEL ASPECT

Abstract. For a fixed imaginary quadratic field, we conjecture that finite-order twists of cyclic base change have density zero among non-CM rational Bianchi newforms of parallel weight two and trivial central character, ordered by the norm of the level. We record the observed decay in the deepest available fixed-field datasets and isolate the counting problem required for a proof.

1. Rational Bianchi newforms

Fix an imaginary quadratic field K. Let $\mathcal{R}_K(X)$ be the set of Galois orbits of cuspidal Bianchi newforms F over K satisfying

$$k_F = 2, \qquad \omega_F = 1, \qquad \mathbb{Q}_F = \mathbb{Q}, \qquad F \text{ non-CM}, \qquad \mathrm{N}\mathfrak{N}_F \le X,$$

where each Galois orbit is counted once. Let $\mathcal{L}_K(X)$ be the subset consisting of forms of the shape

$$F \simeq \mathrm{BC}_{K/\mathbb{Q}}(f) \otimes \psi,$$

where f is a classical newform and ψ is a finite-order Hecke character of K.

Conjecture 1. *For every imaginary quadratic field K,*

$$\#\mathcal{R}_K(X) \longrightarrow \infty \quad \Longrightarrow \quad \lim_{X\to\infty} \frac{\#\mathcal{L}_K(X)}{\#\mathcal{R}_K(X)} = 0.$$

2. Numerical evidence

The available data contain 232143 qualifying non-CM rational orbits over 501 represented fields; 17053 are marked as direct or twisted base change. Because the level coverage is highly nonuniform, the meaningful statistic is fixed-field decay rather than the pooled proportion.

For the five deepest datasets, the cumulative lifted proportion changed as follows:

The final disjoint high-level bins in these five fields contain approximately 1.7%–2.6% lifted forms. All nine fields with substantial data beyond level norm 1000 showed a lower terminal cumulative ratio than at $X = 1000$.

Key words and phrases. Bianchi modular forms, base change, rational Hecke fields, level aspect.

Disc(K)	proportion at $X = 1000$	terminal proportion
-3	0.169231	0.031664
-4	0.130890	0.029200
-7	0.126582	0.027332
-8	0.142045	0.034867
-11	0.163077	0.034765

Table 1. Decay of the lifted proportion in the deepest fixed-field datasets.

3. Why density zero is plausible

Base change and its finite-order twists arise from a lower-dimensional lifting construction. The full Bianchi cohomology should contain genuinely three-dimensional automorphic phenomena that become dominant as the level grows. The observed concentration of lifted forms at very small level and their subsequent decay is consistent with this expectation.

The rationality restriction makes the conjecture substantially harder than a dimension comparison. Complex dimensions weight a Galois orbit by the degree of its Hecke field, whereas $\mathcal{R}_K(X)$ counts only degree-one orbits.

4. A lower bound for the lifted family

The growth assumption in Conjecture 1 is in fact easy to satisfy. Fix a non-CM elliptic curve over $\mathbb{Q}$ and let Π be its cyclic base change to K. Twisting Π by quadratic Hecke characters χ/K preserves parallel weight 2, rational Hecke field, non-CM status and trivial central character. Away from a fixed finite set,

$$\mathfrak{n}(\Pi \otimes \chi) = \mathfrak{n}(\Pi)\mathfrak{f}(\chi)^2.$$

Counting quadratic characters by conductor therefore yields

$$\#\mathcal{L}_K(X) \gg_K X^{1/2}.$$

Consequently $\#\mathcal{R}_K(X) \to \infty$ for every K. This lower bound does not determine the ratio: the full rational family may grow faster, at the same rate, or irregularly.

5. Relation with known results

Dimension formulae for lifted and non-genuine Bianchi subspaces are available at certain squarefree, Galois-stable levels. They concern full complex dimensions and do not count rational Galois orbits. Published finite computations of rational Bianchi forms provide evidence but no varying-level asymptotic. The possibility that twisting a classical form with nonrational coefficient field produces a rational Bianchi form also prevents one from counting the numerator using only rational classical sources.

6. Open difficulties

A proof requires
$$\#\mathcal{L}_K(X) = o(\#\mathcal{R}_K(X)).$$
A plausible upper bound for the numerator is
$$\#\mathcal{L}_K(X) \ll_{K,\varepsilon} X^{1/2+\varepsilon},$$
but this must include ramified finite-order twists and sources whose rationality changes after twisting. The essential missing input is a lower bound for the denominator that grows faster than $X^{1/2}$, or a direct abundance theorem for genuine rational Bianchi forms. Cohomological dimension estimates do not provide such a bound because they do not isolate Hecke-field degree one.

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