UNIT-SIGNATURE RANKS IN EQUAL-DISCRIMINANT CUBIC FIELDS

Abstract. For ordered pairs of distinct totally real S_3-cubic fields with the same discriminant, we conjecture that the two unit-signature ranks are asymptotically independent. The conjecture separates the 2-primary archimedean invariant from the 3-primary ring-class mechanism producing common-discriminant multiplets.

1. Statement of the conjecture

For a positive discriminant D, let $\mathcal{M}_D$ be the set of $\mathbb{Q}$-isomorphism classes of totally real S_3-cubic fields of discriminant D, and put

$$\overrightarrow{\mathcal{P}}(X) = \{(K, K') : K \neq K',\ K, K' \in \mathcal{M}_D \text{ for some } D \leq X\}.$$

For a totally real cubic field K, define

$$s(K) = \dim_{\mathbb{F}_2} \operatorname{im}\big(\operatorname{sgn} : \mathcal{O}_K^\times \longrightarrow \{\pm 1\}^3\big) \in \{1, 2, 3\}.$$

For $s, t \in \{1, 2, 3\}$, let

$$\begin{aligned} u_s(X) &= \Pr_{(K,K') \in \overrightarrow{\mathcal{P}}(X)} \big[s(K) = s\big], \\ b_{s,t}(X) &= \Pr_{(K,K') \in \overrightarrow{\mathcal{P}}(X)} \big[s(K) = s,\ s(K') = t\big]. \end{aligned}$$

Conjecture 1. *Every marginal $u_s(X)$ has positive lower limit, and for all $s, t \in \{1, 2, 3\}$,*

$$\lim_{X \to \infty} \frac{b_{s,t}(X)}{u_s(X) u_t(X)} = 1.$$

2. Numerical evidence

The common-discriminant sample exhibits no large persistent correlation after accounting for the one-field marginal distribution. The most stable cells are those involving signature ranks 2 and 3; the rank-1 cells are much rarer and therefore converge more slowly. Across cumulative cutoffs and in the wider stress sample, the observed joint-frequency matrix was close to the outer product of its marginals, with the largest visible discrepancies concentrated in the sparse rank-1 rows and columns.

The common-discriminant weighting is important. A discriminant supporting m_D fields contributes $m_D(m_D - 1)$ ordered pairs, so large multiplets

Key words and phrases. cubic fields, units, signatures, common discriminants, arithmetic statistics.

receive substantial weight. Numerical agreement therefore tests more than independence in the ordinary one-field family.

3. Heuristic motivation

The multiplicity of totally real cubic fields with one discriminant is controlled by 3-primary class or ring-class data in the common quadratic resolvent. By contrast, the unit-signature rank is an archimedean invariant tied to the 2-Selmer signature map. This separation of primes suggests that conditioning on a large 3-primary multiplet should not change the limiting 2-primary signature distribution.

The conjecture is nevertheless nontrivial. Siblings share the complete discriminant and the same quadratic resolvent, and the pair measure is strongly biased toward resolvents with unusually large 3-torsion. It is therefore necessary to control both arithmetic correlations within a multiplet and heterogeneity between different discriminants.

4. Relation with known heuristics

Dummit–Voight heuristics predict the one-field distribution of unit-signature ranks in totally real S_3-cubic fields. Common-discriminant multiplicity formulas describe the 3-primary mechanism producing the sibling family. Neither framework gives a joint law for two distinct cubic subfields of the same ring-class construction. Existing average results for 2-torsion under local or shape restrictions do not cover the size-biased pair measure appearing here.

5. Open difficulties

Write

$$m_D = \#\mathcal{M}_D, \qquad k_{D,s} = \#\{K \in \mathcal{M}_D : s(K) = s\}.$$

Then the conjecture asks for asymptotics of

$$\sum_{D \le X} k_{D,s} k_{D,t} \quad \text{and} \quad \sum_{D \le X} k_{D,s}(m_D - 1)$$

relative to

$$\sum_{D \le X} m_D(m_D - 1).$$

The diagonal case $s = t$ requires the appropriate factorial correction because $K \neq K'$. A proof would need uniform control of the signature distribution within variable ring-class multiplets and of the tails of the $m_D(m_D - 1)$ weighting. No present theorem supplies such a decorated multiplet count.

References

[1] D. S. Dummit and J. Voight, *The* 2*-Selmer group of a number field and heuristics for narrow class groups and signature ranks of units*, Proc. Lond. Math. Soc. **117** (2018), 682–726.
[2] D. C. Mayer, *Classifying multiplets of totally real cubic fields*, Int. J. Number Theory **18** (2022), 813–852.
[3] M. Bhargava and I. Varma, *The mean number of* 3*-torsion elements in the class groups and ideal groups of quadratic orders*, Proc. Lond. Math. Soc. **112** (2016), 235–266.