CLASS-NUMBER DIVISIBILITY IN EQUAL-DISCRIMINANT CUBIC FIELDS

Abstract. We study pairs of distinct totally real S_3-cubic fields with the same discriminant. For every prime $\ell \geq 5$, we conjecture that divisibility of the two class numbers by ℓ is asymptotically independent under the natural ordered-pair weighting. The exclusion of $\ell = 3$ is essential and reflects the 3-primary class-field-theoretic mechanism producing common-discriminant multiplets.

1. Common-discriminant multiplets

For a positive discriminant D, let $\mathcal{M}_D$ denote the set of $\mathbb{Q}$-isomorphism classes of totally real S_3-cubic fields of discriminant D. Define the set of ordered sibling pairs

$$\overrightarrow{\mathcal{P}}(X) = \{(K, K') : K \neq K',\ K, K' \in \mathcal{M}_D \text{ for some } D \leq X\}.$$

For a prime $\ell \geq 5$, put

$$u_\ell(X) = \frac{\#\{(K, K') \in \overrightarrow{\mathcal{P}}(X) : \ell \mid h_K\}}{\#\overrightarrow{\mathcal{P}}(X)},$$

$$b_\ell(X) = \frac{\#\{(K, K') \in \overrightarrow{\mathcal{P}}(X) : \ell \mid h_K \text{ and } \ell \mid h_{K'}\}}{\#\overrightarrow{\mathcal{P}}(X)}.$$

Conjecture 1. *For every prime $\ell \geq 5$,*

$$\#\overrightarrow{\mathcal{P}}(X) \longrightarrow \infty, \qquad \liminf_{X\to\infty} u_\ell(X) > 0,$$

and

$$\lim_{X\to\infty} \frac{b_\ell(X)}{u_\ell(X)^2} = 1.$$

2. Numerical evidence

In a primary sample of 414436 ordered pairs, the results for the first few primes were as follows.

For $\ell = 5$, independence predicts about 37.10 joint ordered observations, compared with 34 observed. For $\ell = 7$, the six ordered observations correspond to only three unordered double events, so the deviation has little statistical significance. The data for $\ell \geq 11$ are too sparse for a meaningful test.

Key words and phrases. cubic fields, class numbers, common discriminants, Cohen–Martinet heuristics.

ℓ	endpoint events	joint ordered events	u_ℓ	b_ℓ/u_ℓ^2
5	3921	34	0.00946105	0.9165
7	1335	6	0.003221	1.3952
11	234	0	0.000565	—

Table 1. Class-number divisibility among ordered sibling pairs.

The contrast at $\ell = 3$ is striking: every tested range had perfect synchronization between siblings. This confirms that the restriction $\ell \geq 5$ is arithmetic rather than cosmetic.

3. Heuristic explanation

Sibling fields share their discriminant, quadratic resolvent and all discriminant-level ramification data. Their existence and multiplicity are controlled by 3-primary ring-class information in the common quadratic resolvent. For primes $\ell \geq 5$, which are coprime to $|S_3|$, Cohen–Martinet heuristics suggest that the ℓ-primary class groups should behave as fresh random data for each sibling. Conjecture 1 asserts that this remains true even under the strong conditioning that the complete discriminants agree.

4. A proved part of the statement

The divergence of the number of sibling pairs follows from known lower bounds for real quadratic fields with large 3-rank. If a real quadratic field of discriminant d has 3-rank r, then its unramified cyclic cubic extensions give

$$\frac{3^r - 1}{2}$$

distinct totally real cubic fields of discriminant d. Quantitative results producing infinitely many quadratic fields with $r \geq 4$ therefore imply

$$\#\overrightarrow{\mathcal{P}}(X) \gg \frac{X^{1/30}}{\log X}.$$

This proves the first assertion of Conjecture 1, but gives no control of good-prime divisibility in the cubic class numbers.

5. Open difficulties

Let

$$m_D = \#\mathcal{M}_D, \qquad k_D = \#\{K \in \mathcal{M}_D : \ell \mid h_K\}.$$

Then

$$N(X) = \sum_{D \leq X} m_D(m_D - 1),$$
$$A(X) = \sum_{D \leq X} k_D(m_D - 1),$$

$$B(X) = \sum_{D \le X} k_D(k_D - 1).$$

The desired conclusion is equivalent to

$$\liminf_{X\to\infty} \frac{A(X)}{N(X)} > 0, \qquad \frac{B(X)N(X)}{A(X)^2} \longrightarrow 1.$$

This is a second-factorial-moment problem with the strongly size-biased weight $m_D(m_D - 1)$. Conditional independence at each fixed D would not suffice if the marginal probability varied with D; uniformity in the quadratic resolvent, conductor and multiplet size is also required.

References

[1] D. C. Mayer, *Multiplicities of dihedral discriminants*, Math. Comp. **58** (1992), 831–847.

[2] D. C. Mayer, *Classifying multiplets of totally real cubic fields*, Int. J. Number Theory **18** (2022), 813–852.

[3] H. Cohen and J. Martinet, *Class groups of number fields: numerical heuristics*, Math. Comp. **48** (1987), 123–137.

[4] W. Wang and M. M. Wood, *Moments and interpretations of the Cohen–Lenstra–Martinet heuristics*, Comment. Math. Helv. **96** (2021), 339–387.