MONOGENICITY AND UNIT-SIGNATURE RANK IN TOTALLY REAL CUBIC FIELDS

Abstract. We conjecture that monogenicity and unit-signature rank are asymptotically independent in the discriminant-ordered family of totally real non-Galois cubic fields. We present the numerical ratios motivating the conjecture and discuss why existing results on monogenic fields and on narrow class groups do not presently imply it.

1. Statement of the conjecture

For $X > 0$, let

$$F(X) = \left\{K : \ [K : \mathbb{Q}] = 3,\ r_2(K) = 0,\ \mathrm{Gal}(\widetilde{K}/\mathbb{Q}) \simeq S_3,\ |\operatorname{Disc} K| \leq X\right\},$$

where fields are counted up to $\mathbb{Q}$-isomorphism. Let

$$M(X) = \{K \in F(X) : \mathcal{O}_K = \mathbb{Z}[\alpha] \text{ for some } \alpha \in \mathcal{O}_K\}$$

be the subfamily of monogenic fields.

For a totally real cubic field K, define its unit-signature rank by

$$\mathrm{sgnrk}(K) = \dim_{\mathbb{F}_2} \mathrm{im}(\mathrm{sgn} : \mathcal{O}_K^\times \longrightarrow \{\pm 1\}^3).$$

For $s \in \{1, 2, 3\}$, put

$$F_s(X) = \{K \in F(X) : \mathrm{sgnrk}(K) = s\}, \qquad M_s(X) = M(X) \cap F_s(X).$$

Conjecture 1. *For each* $s \in \{1, 2, 3\}$,

$$\lim_{X\to\infty} \frac{\#M_s(X)\,\#F(X)}{\#M(X)\,\#F_s(X)} = 1.$$

Equivalently, the limiting distribution of the unit-signature rank is unchanged after conditioning on monogenicity.

2. Numerical evidence

The proportions of signature ranks $1, 2, 3$ among monogenic fields at several discriminant cutoffs were as follows.

These proportions drift toward the Dummit–Voight full-family prediction

$$(0.019097,\ 0.618304,\ 0.362599).$$

At the largest apparently contiguous cutoff, $X = 3375000$, the data contain

$$\#F(X) = 193179, \qquad \#M(X) = 75689.$$

Key words and phrases. monogenic cubic fields, unit signatures, narrow class groups, arithmetic statistics.

X	rank 1	rank 2	rank 3
100000	0.51%	56.87%	42.62%
500000	0.76%	60.03%	39.22%
1000000	1.05%	60.53%	38.42%
2000000	1.29%	61.31%	37.41%

TABLE 1. Unit-signature ranks among monogenic fields.

For $s = 1, 2, 3$, the pairs $(\#F_s, \#M_s)$ are

$$(1739, 1044), \qquad (110348, 46842), \qquad (81092, 27803),$$

giving the finite ratios

$$1.532244, \qquad 1.083423, \qquad 0.875066.$$

Thus convergence to 1 is not yet visible, especially in the rare signature-rank-one stratum.

3. Arithmetic motivation

Unit signatures are controlled by the archimedean part of the 2-Selmer signature map, whereas monogenicity is the global integral condition that the binary cubic index form represent ± 1. These mechanisms are different enough that asymptotic independence is plausible. At the same time, monogenicity is known to alter certain 2-class-group statistics, so the conjecture is not a formal consequence of existing heuristics.

The fieldwise exact sequence relating signatures, narrow class groups and ordinary class groups gives

$$\frac{h_K^+}{h_K} = 2^{3-\mathrm{sgnrk}(K)}.$$

This identity explains the relation with narrow class groups but does not determine the distribution of the signature rank inside the monogenic subfamily.

4. Relation with known results

Dummit and Voight conjecture the unconditioned distribution of unit-signature ranks in totally real S_3-cubic fields. Results on monogenized cubic fields compute moments of ordinary and narrow 2-torsion under a different ordering, in which a chosen generator is part of the object. They do not count each monogenic field once by field discriminant. Moreover, while a positive proportion of cubic fields is known to be nonmonogenic, no asymptotic formula is known for the number of monogenic cubic fields of bounded discriminant.

5. Open difficulties

A proof of Conjecture 1 requires the correlation estimate

$$\#M_s(X)\#F(X) - \#M(X)\#F_s(X) = o\big(\#M(X)\#F_s(X)\big).$$

A stronger sufficient statement would be the existence of common constants p_s such that

$$\#F_s(X) \sim p_s\#F(X), \qquad \#M_s(X) \sim p_s\#M(X).$$

Neither asymptotic is presently available. The difficulty is to control simultaneously the global index-form equation defining monogenicity and the archimedean 2-Selmer condition defining the signature rank.

References

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