A GEOMETRIC PRODUCT LAW FOR CYCLOTOMIC IWASAWA INVARIANTS

Abstract. We propose a rank-conditioned distribution for ordinary and signed cyclotomic Iwasawa λ-invariants of elliptic curves over $\mathbb{Q}$. After removing the zero forced by Mordell–Weil rank, the remaining half-difference is conjectured to be geometrically distributed, with asymptotic independence across primes and between the two signed invariants at a supersingular prime.

1. The conditioned family

Let S be a finite set of primes $p \geq 11$, let $r \geq 1$, and prescribe for each $p \in S$ either ordinary or supersingular reduction, denoted by a type function τ. Let $F_{S,r,\tau}(X)$ be the set of $\mathbb{Q}$-isogeny classes $[E]$ of non-CM elliptic curves $E/\mathbb{Q}$ satisfying $N_E \leq X$ and $\operatorname{rank} E(\mathbb{Q}) = r$, together with the following conditions at every $p \in S$:

(1) E has good reduction at p of the prescribed type;
(2) the residual representation on $E[p]$ is surjective;
(3) $p \nmid \operatorname{Tam}(E)$;
(4) the relevant analytic cyclotomic μ-invariant vanishes;
(5) in the ordinary case, $p \nmid \# E(\mathbb{F}_p)$.

Put

$$I_\tau = \{(p,0) : \tau(p) = \mathrm{ord}\} \cup \{(p,+),(p,-) : \tau(p) = \mathrm{ss}\}.$$

For ordinary p, define

$$K_{p,0}(E) = \frac{\lambda_p(E) - r}{2},$$

and for supersingular p, using the Pollack signed normalization, define

$$K_{p,\pm}(E) = \frac{\lambda_p^{\pm}(E) - r}{2}.$$

Conjecture 1. *Assume that $\# F_{S,r,\tau}(X) \to \infty$. Then, for every vector $\mathbf{k} = (k_{p,s})_{(p,s)\in I_\tau} \in \mathbb{Z}_{\geq 0}^{I_\tau}$,*

$$\lim_{X\to\infty} \frac{\#\{[E] \in F_{S,r,\tau}(X) : K_{p,s}(E) = k_{p,s} \text{ for all } (p,s) \in I_\tau\}}{\# F_{S,r,\tau}(X)} = \prod_{(p,s)\in I_\tau} (1-p^{-1})p^{-k_{p,s}}.$$

In particular, all the coordinates are asymptotically independent, including the two signs at a single supersingular prime.

Key words and phrases. elliptic curves, Iwasawa invariants, signed p-adic L-functions, arithmetic statistics.

2. Numerical evidence

The available Iwasawa data cover conductor at most 149996. After passing to $\mathbb{Q}$-isogeny classes, the rank-one ordinary sample at $p = 11$ contained 204573 classes. The observed counts for $K_{11,0} = 0, 1, 2$ and $K_{11,0} \geq 3$ were

$$186137, \qquad 16792, \qquad 1427, \qquad 217,$$

close to the geometric probabilities 10/11, 10/121, 10/1331 and the remaining tail.

For $p = 13$ supersingular and rank 1, a sample of 23707 classes gave

$(K_{13,+}, K_{13,-})$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
count	20240	1516	1566	127.

These figures are close to the prediction from two independent geometric variables. Same-prime signed covariances were small, and the largest absolute observed covariance was approximately 0.00211.

For the pair of primes 11 and 13 in rank 1, the observed probabilities that all relevant coordinates were minimal were

$$0.839467, \quad 0.778247, \quad 0.763577, \quad 0.701206$$

for the ordinary/ordinary, ordinary/supersingular, supersingular/ordinary and supersingular/supersingular strata. The corresponding product predictions were

$$0.839161, \quad 0.774610, \quad 0.762873, \quad 0.704191.$$

The same comparison was performed for all 55 pairs among the primes from 11 through 47 for which reliable data were available.

3. Random-coefficient heuristic

After extracting the rank-forced factor T^r from a self-dual p-adic L-function with $\mu = 0$, functional-equation symmetry forces the remaining Weierstrass degree to have the same parity as r. Write the symmetry-compatible coefficients as

$$a_0, a_1, a_2, \ldots$$

in degrees $r, r+2, r+4, \ldots$. If their reductions modulo p behaved as independent uniform elements of $\mathbb{F}_p$, then

$$K = \min\{j : a_j \neq 0\}$$

would satisfy

$$\Pr(K = k) = (1 - p^{-1})p^{-k}.$$

Independence of the coefficient systems would yield the product law in Conjecture 1. The local exclusions in the definition of the family remove deterministic contributions, but they do not prove this randomness.

4. Relation with known results

Ordinary and signed Iwasawa theory provide the parity constraints and the two supersingular invariants. Existing arithmetic-statistical results give bounds or positive-density statements for prescribed Iwasawa invariants, and recent work gives criteria for the minimal rank-one event $\mu = 0$, $\lambda = 1$. None of these results proves the full geometric tail, fixed-rank conductor-ordered limits, cross-prime independence, or independence of the two signed invariants at one supersingular prime.

The latter is particularly delicate: the two signed p-adic L-functions have different Coleman maps, but share the underlying Mordell–Weil lattice and may share factors arising from Tate–Shafarevich or fine Selmer groups.

5. Open difficulties

Even the subcase consisting of one ordinary prime, rank 1, and $K = 0$ would require proving a conductor-ordered density exactly equal to $1 - p^{-1}$. The full statement asks for simultaneous equidistribution of all symmetry-compatible coefficients, conditional on rank, residual surjectivity, vanishing μ-invariant and the stated local restrictions. No theorem presently controls these correlations, either across different primes or between the two signed theories at one prime.

References

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