# RANK-CONDITIONED TAILS OF LOCAL TAMAGAWA NUMBERS

Abstract. We formulate a conjectural power-law distribution for local Tamagawa numbers at marked split multiplicative primes of semistable elliptic curves over $\mathbb{Q}$, ordered by conductor and conditioned by Mordell–Weil rank and the sign of the minimal discriminant. We record the numerical evidence motivating the conjecture and explain the gap between the proposed conductor-ordered statement and known fixed-prime, height-ordered local density results.

## 1. Statement of the conjecture

Fix an integer $r \geq 0$ and a sign $\sigma \in \{-1, +1\}$. For $X > 0$, let

$$\begin{aligned}\mathcal{S}_{r,\sigma}(X) = \{(E, q) : N_E \leq X,\ E/\mathbb{Q} \text{ semistable and non-CM},\\ \operatorname{rank} E(\mathbb{Q}) = r, \quad [E]_{\mathbb{Q}\text{-isog}} = \{E\},\\ \operatorname{sgn}(\Delta_{\min}(E)) = \sigma, \quad E \text{ has split multiplicative reduction at } q\}.\end{aligned}$$

Here elliptic curves are counted up to $\mathbb{Q}$-isomorphism, while the prime $q$ is marked. Thus a curve with several split multiplicative primes contributes once for each such prime.

For a prime $\ell$, define, whenever the limit exists,

$$\delta_{r,\sigma}(\ell) = \lim_{X\to\infty} \frac{\#\{(E, q) \in \mathcal{S}_{r,\sigma}(X) : c_q(E) = \ell\}}{\#\mathcal{S}_{r,\sigma}(X)}.$$

**Conjecture 1.** *For every $r \geq 0$, every $\sigma \in \{-1, +1\}$ and every prime $\ell$, the density $\delta_{r,\sigma}(\ell)$ exists. Moreover,*

$$\lim_{\substack{\ell\to\infty \\ \ell \text{ prime}}} \ell^2 \delta_{r,\sigma}(\ell) = r + 1.$$

The order of limits is part of the statement: one first lets the conductor bound $X$ tend to infinity for fixed $\ell$, and only afterwards lets $\ell$ tend to infinity through the primes.

## 2. Numerical evidence

In the complete conductor range $N_E \leq 150000$, the marked split-multiplicative samples contained 62395, 116608 and 41390 pairs in ranks 0, 1 and 2, respectively. The following table gives the observed values of $\ell^2 \Pr(c_q(E) = \ell)$ after pooling both signs of the minimal discriminant.

*Key words and phrases.* elliptic curves, Tamagawa numbers, split multiplicative reduction, conductor ordering, Mordell–Weil rank.

| rank | $\ell = 5$ | $\ell = 7$ | $\ell = 11$ | $\ell = 13$ |
|---|---|---|---|---|
| 0 | 1.162 | 1.253 | 1.187 | 1.181 |
| 1 | 1.678 | 1.986 | 2.018 | 1.935 |
| 2 | 1.883 | 2.388 | 2.850 | 2.683 |

Table 1. Finite-conductor estimates for the scaled local Tamagawa frequency.

Six conductor windows gave comparable values. Splitting by the sign of $\Delta_{\min}$ preserved the same rank dependence. At $\ell = 11$, the negative- and positive-sign estimates were respectively 1.237 and 1.032 in rank 0, 2.028 and 1.992 in rank 1, and 2.835 and 2.879 in rank 2.

A wider check in the uniform range $N_E \leq 500000$ involved 751417 marked pairs. The predicted ordering by rank persisted across five cumulative conductor cutoffs. For $\ell$ beyond roughly 17–23, the finite-cutoff estimates decrease; this is consistent with the fact that a fixed conductor cutoff suppresses very large valuations of the minimal discriminant.

## 3. Heuristic motivation

At a split multiplicative prime of Kodaira type $I_n$, Tate's algorithm gives

$$c_q(E) = n = v_q(\Delta_{\min}(E)).$$

Thus Conjecture 1 is a statement about the tail of a marked local discriminant valuation. Conductor ordering records the presence of the bad prime $q$ but does not directly charge for the size of $v_q(\Delta_{\min})$, making a polynomial tail plausible. The data suggest that the exponent is stable while conditioning on rank changes the leading coefficient. Marking $q$ removes variation in the number of bad places, and restricting to singleton rational isogeny classes avoids transfers of Tamagawa factors within an isogeny class.

Rank-conditioned Selmer heuristics offer a possible explanation for the factor $r + 1$: conditioning on a larger Mordell–Weil rank may size-bias local arithmetic data. At present this remains only a heuristic analogy.

## 4. Relation with known results

Known local density formulae at a fixed residue prime are in a different regime. For fixed $p$, the density of minimal Weierstrass models of type $I_m$ decays exponentially in $m$. Global results on products of Tamagawa numbers are likewise generally height-ordered. Existing conductor-ordered counting theorems allow only restricted families and finitely many prescribed local conditions; they do not impose exact Mordell–Weil rank, a marked varying prime, or a large-prime uniformity statement of the form above.

## 5. Open difficulties

A proof would require asymptotics for

$$A_{r,\sigma}(X) = \#\mathcal{S}_{r,\sigma}(X), \qquad A_{r,\sigma,\ell}(X) = \#\{(E,q) \in \mathcal{S}_{r,\sigma}(X) : v_q(\Delta_{\min}(E)) = \ell\},$$

first for each fixed $\ell$ and then uniformly enough in $\ell$ to obtain $\delta_{r,\sigma}(\ell) \sim (r+1)/\ell^2$. No available theorem gives such simultaneous control under exact-rank conditioning. The universal quantifier in $r$ is also far beyond the numerical range: the available data are statistically meaningful only for $r = 0, 1, 2$.

## References

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